Spectral Winding and Quantum Anomaly

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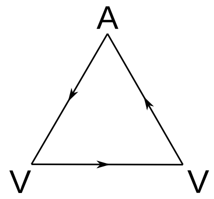
Based on a work in progress

1. Introduction

Anomaly

- Chiral anomaly (called also gauge anomaly, ABJ anomaly, non-Abelian anomaly, ... according to different emphases) arises when symmetries of quantum field theories (QFT) "clashes".
- Some symmetries are mutually inconsistent and (at least) one of them becomes broken in quantum theory .
- Anomaly is discovered through analysis of a fermion loop diagram (the triangular diagram) Adler; Bell, Jackiw; Bardeen '69 following earlier computations Fukuda, Miyamoto;Steinberger '49



Anomaly allows many different point of views

- Remarkably, discoveries of new view points (interpretations and rephrasings) of anomaly are continuing over the years.
  - relation to Atiyah-Singer index theorems Jackiw, Rebbi; Nielsen, Schroer '77
  - interpretation via Jacobian in the path integral formulation Fujikawa '79
  - Anomaly descent relations Zumino; Stora; Baulieu '83
  - Anomaly inflow Callan, Harvey '85
  - Connection to quantum Hall effect late '80s?
  - Realisation in lattice gauge theory via Ginsparg-Wilson relation Lüscher '98
  - etc, etc

The winding number of $\det D$

- Anomaly is closely related to (can be detected by) "the winding number of a family of Dirac operators". (Atiyah, Singer;Alvarez-Gaume, Ginsparg; Sumitani; Alvarez, Singer, Zumino '84)
- One considers an one-parameter family of linear operators (called Dirac operators) $D(\theta)$, $D(0) = D(2\pi)$, associated with the theory.
- The determinant of $D(\theta)$, $\det D(\theta) \in \mathbb{C}$ defines a closed loop on the complex plane. One considers the winding number around 0 of the complex plane of this loop as θ is varied from 0 to 2π.
- Non-zero winding number of $\det D$ implies the existence of anomaly .

Main points of this talk

1. I propose the concept of collective winding of the eigenvalue spectrum (which I call the "spectral winding"). In some cases, eigenvalues may be thought to have fractional winding numbers. The spectral winding is a refined version of the winding number of $\det D$, which detects anomaly.
2. I have constructed examples of QFT (or Dirac operator) exhibiting spectral winding (including the case exhibiting fractional winding number).
3. The case with fractional winding number may lead to a new way of producing 2D chiral CFT from 4D field theory using "vortex-like" configuration (via anomaly inflow).

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3. Spectral Winding
4. Examples
5. Vortex-like configuration in 4D theory and 2D chiral CFT: case with fractional winding numbers
6. Summary and discussion

2. Anomaly and winding number of determinant

A definition of (perturbative) anomaly

- Consider a quantum field theory in background fields B with classical symmetry transformation $B \to B + \delta B$.
- If the (phase of) the partition function $Z[B]$ changes under the (infinitesimal) symmetry transformation , in a not too trivial way ("cannot be absorbed by adding the local terms"),

$$Z[B + \delta B] = e^{i \int a(B, \delta B)} Z[B], \quad a \neq \delta(b(B, \partial B))$$

  we have an anomaly (Bardeen '69) .
- NB: This definition encompasses both Abelian (ABJ) anomaly and non-Abelian anomaly.
- NB: Using this definition, the anomaly is not restricted to the massless theories; one can consider masses as part of the background fields B.

winding number of $\det D$

- For many theories of interest, the partition function Z can be written (formally) as the determinant of a Dirac operator $D[B]$
$$Z[B] = \det D[B]$$
NB: D includes the contribution from the mass term.

- Anomaly can be detected (characterised) by the winding number of $\det D$.
    - Find an one-parameter family of (finite) gauge transformations $g(\theta)$, such that $B^{g(0)} = B^{g(2\pi)}$ and $\det D[B^{g(\theta)}] \in \mathbb{C}$ has non-zero winding number around 0 in the complex plane. (B^g: the gauge transformation of B by g.)
    - Then anomaly must exist. ($Z[B]$ must also change under infinitesimal gauge transformation.)

The specific class of theories we consider

- To be specific, we focus on N Dirac fermions in even dimensional Euclidean flat spacetime coupled with background fields $B = (M(x), A_{\pm\mu}(x))$
- The Euclidean action is

$$S = \int d^d x \begin{bmatrix} \overline{\psi}_- & \overline{\psi}_+ \end{bmatrix} \begin{bmatrix} M & \sigma^\mu D_{-\mu} \\ \tilde{\sigma}^\mu D_{+\mu} & M^\dagger \end{bmatrix} \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix}$$

- $\overline{\psi}_+$ ($\overline{\psi}_-$): N copies of positive (negative) chirality spinors
- $M(x)$: a complex $N \times N$ matrix (the mass parameter which is allowed to be spacetime dependent)
- $A_{\pm\mu}(x)$: $N \times N$ anti-hermitian matrices; gauge fields acting on $\overline{\psi}_\pm$
- $\sigma^\mu, \tilde{\sigma}^\nu$ are basically the Dirac γ-matrices.

Classical symmetry

- The action

$$S = \int d^dx \begin{bmatrix} \overline{\psi}_- & \overline{\psi}_+ \end{bmatrix} \begin{bmatrix} M & \sigma^\mu D_{-\mu} \\ \tilde{\sigma}^\mu D_{+\mu} & M^\dagger \end{bmatrix} \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix}$$

has (in addition to the "Lorentz symmetry" $\mathrm{SO(d)}$), the $\mathrm{U(N)} \times \mathrm{U(N)}$ classical gauge symmetry parametrised by $g = (U_+(x), U_-(x))$

$$S[\psi^g, \overline{\psi}^g; A^g, M^g] = S[\psi, \overline{\psi}; A, M]$$

with the symmetry transformations,

$$\psi_\pm \to \psi^g_\pm = U_\pm \psi_\pm, \quad \overline{\psi}_\pm \to \overline{\psi}^g_\pm = \overline{\psi}_\pm U_\pm^{-1}$$

$$M \to M^g = U_- M U_+^{-1}$$

$$A_{\pm\mu} \to A^g_{\pm\mu} = U_\pm A_\mu U_\pm^{-1} - \partial_\mu U_\pm U_\pm^{-1}$$

The partition function and the Dirac operator

- The partition function $Z[B]$ of the theory can formally be defined as the determinant of the operator D ,
$$Z[B] = \int \mathcal{D}\psi\mathcal{D}\overline{\psi}e^{-S[\psi,\overline{\psi};B]} = \det D[B]$$
where
$$D = \begin{bmatrix} M & \sigma^\mu D_{-\mu} \\ \tilde{\sigma}^\mu D_{+\mu} & M^\dagger \end{bmatrix}$$
- NB: This model (and its anomaly) we are considering includes theory of chiral fermion (and its anomaly) as a special case; put, e. g. $M = 0$ and $A_- = 0$; Then Z factorises, $Z = \det D = \det \tilde{\sigma}^\mu D_{+\mu} \times \det \sigma^\mu \partial_\mu$ the first factor is Z of the chiral fermion and the second factor is an unimportant constant (i. e. does not depends on the background.) Leutwyler; Alvarez-Gaume, Ginsparg '84

Anomaly of the model

- It is well-known that the $\mathrm{U(N)} \times \mathrm{U(N)}$ symmetry is anomalous
$$Z[B+\delta B] = e^{i\int a(B,\delta B)} Z[B], \quad a \neq \delta(b(B,\partial B))$$
and the anomaly $a(B,\delta B)$ is explicitly computed Bardeen '69, ...

- In particular, the "vector-like gauge transformation" satisfying $U_+ = U_-$ and the "axial-vector like gauge transformation" satisfying $U_+ = U_-^{-1}$ are incompatible.

- NB: Recently, the computation of the anomaly is revisited and it is shown that it fits nicely with the framwork of Quillen's superconnection Cordova, Freed, Lam, Seiberg '19; Kanno, Sugimoto '22

3. Spectral Winding

Determinant vs Eigenvalue spectrum

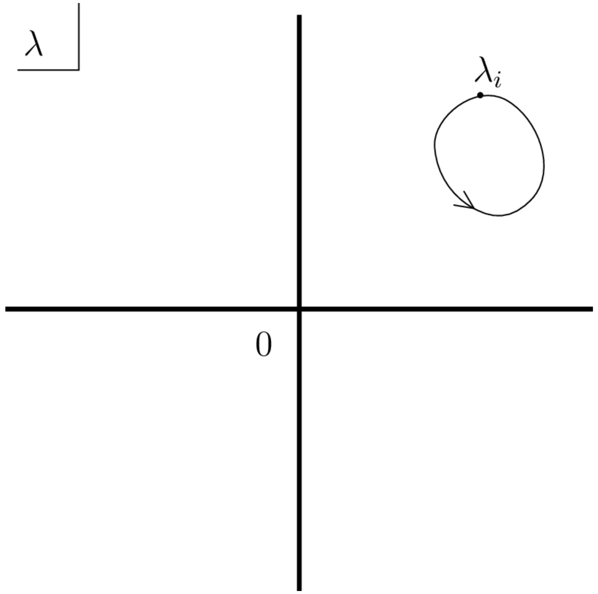
- $\det D$ can be defined as the UV regularised product of eigenvalues,
  $$Z = \det D[B(\theta)] = \prod_i f(\lambda_i)$$
  where
  $$\{\lambda_1, \lambda_2, \dots\}$$
  is the eigenvalue spectrum of D and $f(\lambda)$ is a cutoff function (Λ: cutoff scale)
  $$f(\lambda) = \begin{cases} \lambda & (\lambda \ll \Lambda) \\ 1 & (\lambda \gg \Lambda) \end{cases}$$
- Hence, the winding number of $\det D(\theta)$ can be understood by studying the evolution of the eigenvalue spectrum $\{\lambda_1, \lambda_2, \dots\}$ as $\theta : 0 \to 2\pi$. The latter contains more information.

Spectral winding

- For a closed loop in the space of the backhround field $B(\theta)$, $B(0) = B(2\pi)$, the eigenvalue spectrum $\{\lambda_1, \lambda_2, \ldots\}$ of $D[B(\theta)]$ should go back to itself when $\theta : 0 \to 2\pi$.
- However, each individual eigenvalue may not. There are roughly three patterns:
    i. No winding
    ii. Usual winding
    iii. Collective winding with fractional winding numbers
- Possibilities i. ii. appears e. g. in Alvarez-Gaume, Ginsparg '84 . It seems that the possibility iii. has not been considered previously.

Pattern i. (No winding)

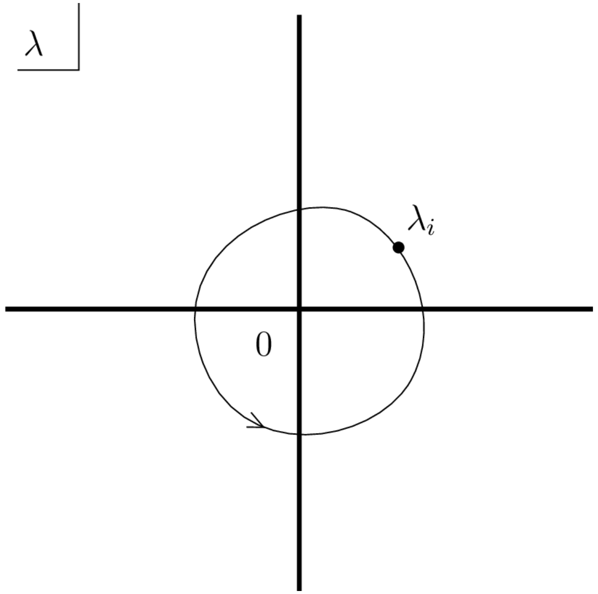
- $\lambda_i(\theta)$ returns to its original position $\lambda_i(0)$ without encircling the origin of the complex plane.



- This should be the typical case for large λ_i ; the eigenvalue should not be affected very much by the background fields and their variations, hence by θ.

Pattern ii. (Usual winding)

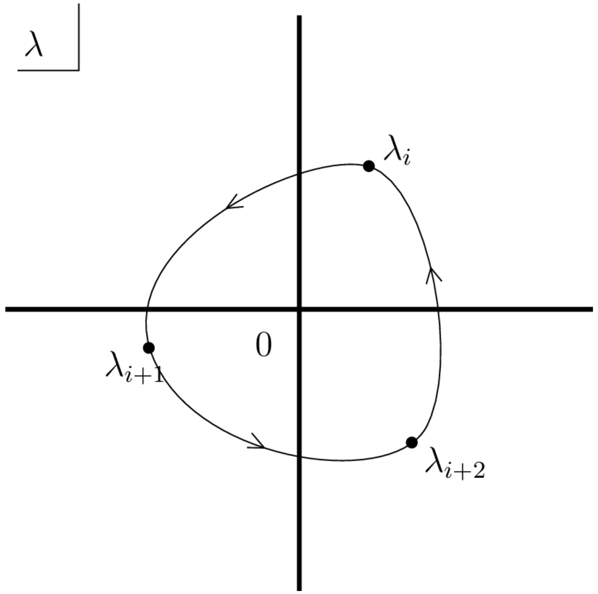
- $\lambda_i(\theta)$ returns to its original position $\lambda_i(0)$ after encircling the origin one or more times.



- In this case the behaviour of λ_i is characterised by a single non-zero winding number.

Pattern iii. (Collective winding with fractional winding numbers)

- λ_i is mapped into another eigenvalue when θ is varied from 0 to 2π.
- For example, λ_i; $\lambda_i(2\pi) = \lambda_{i+1}(0), \lambda_{i+1}(2\pi) = \lambda_{i+2}(0), \lambda_{i+2}(2\pi) = \lambda_i(0)$.



- One may call these three eigenvalues to have "fractional" winding number $1/3$.

Robustness of spectral winding , analogy to spectral flow

- Large eigenvalues will not contribute to the winding number. Only finite numbers of low-lying eigenvalues contribute. "Anomaly is an IR phenomenon (can be understood from the low-energy physics") ('t Hooft '80) .
- Thinking in terms of spectral winding should be thus more well-defined and robust compared to thinking in terms of the determinant of the winding number of $\det D$.
- Similar robustness of the spectral flow (Atiyah, Patodi, Singer '76) of Dirac operator with real eigenvalues is recently emphasised by Fukaya, Onogi, Yamaguchi '17; Fukaya, Furuta, Matsuo, Onogi, Yamashita, Yamaguchi '20; ....
- The spectral winding (for $\lambda \in \mathbb{C}$) may be considered as the analog of the spectral flow (for $\lambda \in \mathbb{R}$).

4. Examples

Eigenvalue Equation

- The eigenvalue equations for our setup are
$$D\begin{bmatrix}\psi_+\\ \psi_-\end{bmatrix}=\begin{bmatrix}M & \sigma^\mu D_{-\mu}\\ \tilde{\sigma}^\mu D_{+\mu} & M^\dagger\end{bmatrix}\begin{bmatrix}\psi_+\\ \psi_-\end{bmatrix}=\lambda\begin{bmatrix}\psi_+\\ \psi_-\end{bmatrix}$$
i. e.
$$\begin{aligned}\sigma^\mu D_{-\mu}\psi_- &= (\lambda - M)\,\psi_+,\\ \tilde{\sigma}^\mu D_{+\mu}\psi_+ &= \left(\lambda - M^\dagger\right)\psi_-.\end{aligned}$$

- Preserves "vector like" part (satisfying $U_+ = U_-$) but breaks "axial-vector" part (satisfying $U_+ = U_-^{-1}$) of $\mathrm{U(N)}\times\mathrm{U(N)}$ symmetry,
$$\psi_\pm \to U_+\psi_\pm, M \to U_- M U_+^{-1}$$

- This is as it should be. We know $\mathrm{U}(N)\times\mathrm{U}(N)$ symmetry of theory is anomalous .

Simplest Example (1)

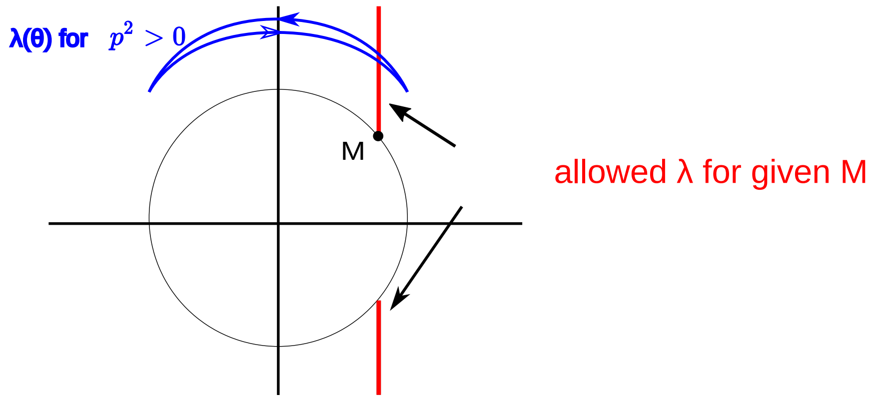
- Consider $N = 1$, $A_+ = A_-$, $M = \mathrm{const.} \in \mathbb{C}$
- The eigenvalue equation can be recast into a 2nd order equation. (Dirac equation ~ square root of the Laplace equation)
  $$\tilde{\sigma}^\nu D_\nu \sigma^\mu D_\mu \psi_- = -p^2 \psi_-$$
  with
  $$-p^2 = (\lambda - M)\left(\lambda - M^\dagger\right) = \lambda^2 - \left(M + M^\dagger\right)\lambda + M M^\dagger$$
- The eigenvalue problem reduces to the hermitian eigenvalue problem with eigenvalue $-p^2 \leq 0$.
  Note that the 2nd order operator does not contain M.

Simplest Example (2)

- The eigenvalue λ of D is related to the eigenvalue $-p^2$ of the (massless) hermitian operator $\tilde{\sigma}^\nu D_\nu \sigma^\mu D_\mu$, $p^2 \geq 0$.

$$\lambda = \mathrm{Re}\, M \pm i\sqrt{p^2 + (\mathrm{Im}\, M)^2}$$

- Consider a $U_+ = U_-^{-1} = e^{\frac{\theta}{2}}$, $M = me^{i\theta}$, $\theta : 0 \to 2\pi$.



- The eigenvalues with $p^2 > 0$ cannot contribute to the winding number.

Simplest Example (3)

- $p^2 = 0$ means $\lambda = M$ or $\lambda = M^\dagger$. The original eigenvalue equation leads to
$$\lambda = M \implies \psi_- = 0, \tilde{\sigma}^\mu D_\mu \psi_+ = 0$$
$$\lambda = M^\dagger \implies \psi_+ = 0, \sigma^\mu D_\mu \psi_- = 0$$
- Thus each positive (negative) chirality solution of the massless Dirac operator presents $\lambda = M$ ($\lambda = M^\dagger$) i. e. eigenvalue of D (with non-zero mass) with positive (negative) winding number .
- Reproduces the $U(1)$ ABJ anomaly with the Atiyah-Singer index theorem, $Z[me^{i\theta}, A] = e^{i\mathrm{ch(F)}\theta} Z[m, A] = e^{i(n_+ - n_-)} Z[m, A]$
- Extension to the $U(N)$ theory is easy : consider $A_+ = A_-$, $M = mI_N$

IR regularisation

- To make the argument more precise, one needs an IR regularisation to make the eigenvalue spectrum discrete .
- This can be achieved, for example, by using a position-dependent mass profile.
- I have solved the eigenvalue problem with the IR cutoff for an exactly solvable case (rotationally symmetric $d = 2$ case), and verified that the cutoff does work . (A nice exercise dealing with Bessel functions; extension of the square-well potential in quantum mechanics.)

Examples with fractional winding number $1/q$ (1)

- Set $N = q$. Choose the mass matrix M to be the "clock matrix",

$$M = m \begin{bmatrix} 1 & & & 0 \\ & \omega & & \\ & & \ddots & \\ 0 & & & \omega^{q-1} \end{bmatrix}, \omega = e^{i\frac{2\pi}{q}}, m = \mathrm{const.} \in \mathbb{C}$$

$$A_{+\mu} = A_{-\mu} = \begin{bmatrix} a_\mu & & & 0 \\ & a_\mu & & \\ & & \ddots & \\ 0 & & & a_\mu \end{bmatrix}$$

- Consider a constant axial $U(1)$ rotation

$$U_+ = U_-^{-1} = e^{i\frac{1}{2q}\theta} I_N, M(\theta) = e^{i\frac{\theta}{q}} M(0)$$

Examples with fractional winding number $1/q$ (2)

- Although M is transformed into

$$M(2\pi) = U_+ M U_-^{-1} = \omega M(0) = m \begin{bmatrix} \omega & & & 0 \\ & \omega^2 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

It is equivalent to $M(0)$ by vector-like gauge transformation, which is non-anomalous, by the "shift matrix"

$$M(2\pi) = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & 0 & \ddots & 1 \\ 1 & & & 0 \end{bmatrix} M(0) \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & 0 & \ddots & 1 \\ 1 & & & 0 \end{bmatrix}^{-1}$$

- This gives a closed loop in the space of the background fields B (modulo non-anomalous gauge symmetry).

Examples with fractional winding number $1/q$ (3)

$$M(2\pi) = \omega M(0) = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & 0 & \ddots & 1 \\ 1 & & & 0 \end{bmatrix} M(0) \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & 0 & \ddots & 1 \\ 1 & & & 0 \end{bmatrix}^{-1}$$

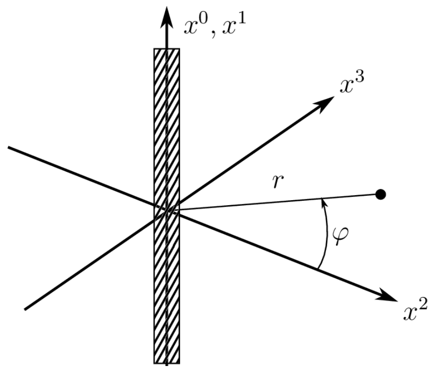
- This gives a closed loop in the space of the background fields B (modulo non-anomalous gauge symmetry).
- NB: One can also write down explicitly $g(\theta) = (U_+(\theta), U_-(\theta))$ satisfying $B^{g(2\pi)} = B^{g(0)}$.
- Varying $\theta : 0 \to 2\pi$ shift the role of the components $1 \to 2$, $2 \to 3$, $\ldots, N \to 1$.
- NB: The use of shift and clock matrices are similar to the so-called non-commutative torus or D-branes put on $\mathbb{Z}_n$ orbifold.

Examples with fractional winding number $1/q$ (4)

- The story now goes roughly as the q copies of the simplest example.
- Since $M = me^{i\frac{\theta}{q}}\mathrm{diag}(1, \omega, \ldots, \omega^{\mathrm{q}-1})$, $\omega = \exp(i\frac{2\pi}{q})$ for each positive chirality solution of the massless Dirac operator, we have q eigenvalues $\lambda_1, \ldots \lambda_q$ with
  $\lambda_1 = me^{i\frac{\theta}{q}}, \lambda_2 = me^{i\frac{\theta+2\pi}{q}}, \ldots$
  and each negative chirality solution of the massless Dirac operator,
  $\lambda_1 = m^\dagger e^{-i\frac{\theta}{q}}, \lambda_2 = m^\dagger e^{-i\frac{\theta+2\pi}{q}}, \ldots$
- We thus have q eigenvalues with fractional winding number $1/q$ $(-1/q)$ for each positive (negative) chirality solution of the massless Dirac operator.
- Again consistent with the ABJ anomaly and the index theorem.

5. Vortex-like configuration in 4D theory and 2D chiral CFT: case with fractional winding numbers

Callan-Harvery anomaly inflow (4D-2D relation)



- Consider a vortex-like configuration in 4D Dirac Fermion theory
$$M(x) = mf(r)e^{in\varphi}$$
$$N = 1, m \in \mathbb{R}, n \in Z, f(0) = 0, f(+\infty) = 1$$
- Solving the 4D Dirac equation (defined on $\mathbb{R}^2 \times \mathbb{R}^2 \setminus \{0\}$) around the configuration one finds $|n|$ solutions of chiral fermions localised on the "vortex". They are "left-moving" (holomorphic in $z = x^0 + ix^1$) for $n > 0$ and "right-moving" (anti-holomorphic) on the vortex for $n < 0$.
- The absence of gauge anomaly in 4D implies a cancellation between the contribution from the "bulk" contribution and the "2D" contribution from the chiral Fermions (anomaly inflow) . Callan, Harvey '85
- This gives evidence that the theory on the vortex are free chiral Fermions.

Vortex-like configuration associated with the fractional winding case (1)

- Consider the configuration

$$M(\varphi) = f(r) e^{i \frac{n}{q} \varphi} \begin{bmatrix} 1 & & & 0 \\ & \omega & & \\ & & \ddots & \\ 0 & & & \omega^{q-1} \end{bmatrix}$$

with $\omega = e^{i \frac{2\pi}{q}}$, $m \in \mathbb{R}, n \in Z, f(0) = 0, f(+\infty) = 1$

- One has a "cut" along $\varphi = 0$,

$$M(\varphi = 2\pi) = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & 0 & \ddots & 1 \\ 1 & & & 0 \end{bmatrix} M(\varphi = 0) \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & 0 & \ddots & 1 \\ 1 & & & 0 \end{bmatrix}^{-1}$$

- Going around the vortex shifts e. g. the diagonal elements $1 \to 2$, $2 \to 3$,

Vortex-like configuration associated with the fractional winding case (2)

- Going around the vortex shifts e. g. the diagonal elements $1 \to 2$, $2 \to 3$, $q \to 1$.
- One can again show that the Dirac equation around the configuration contains $|n|$ solutions of chiral fermions localised on the "vortex".
- This time the field equation can be considered as defined on the product of $\mathbb{R}^2$ (longitudinal direction of the "vortex") and the q-fold cover of $\mathbb{R}^2 \setminus 0$ (transverse direction of the "vortex") .
- This is somewhat reminiscient of the physics of the anyons.

Theory on the vortex-like configuration

- What is this theory on the vortex-like configuration for $q \neq 1$?
- I do not have an answer. Working out the anomaly inflow carefully should give the answer.
- It is tempting to conjecture that the theory on "vortex" may be described by the chiral CFT which is called as the "chiral Luttinger liquid" theory appearing in fractional quantum Hall effect (FQHE) (related to 3D-2D version of the anomaly inflow).
    - It is argued that the "edge mode" of FQHE (with filling fraction $1/q$) is NOT described by the free chiral fermion CFT (which is equivalent to free chiral boson by the bosonisation map $\psi = e^{i\phi}$). Wen '1992
    - The chiral Luttinger liquid is the theory obtained by using $\psi = e^{iq\phi}$.

5. Summary

Summary

1. One can consider the concept of collective winding of the eigenvalue spectrum (which I call the "spectral winding"). In some cases, eigenvalues may be thought to have fractional winding numbers. The spectral winding is a refined version of the winding number of $\det D$, which detects anomaly.

2. One can construct examples of QFT (or Dirac operator) exhibiting spectral winding (including the case exhibiting fractional winding number). The construction uses "shift" and "clock" matrices and is analogous to the non-commutative torus and D-branes on a orbifold.

3. The case with fractional winding number may lead to a new way of producing 2D chiral CFT from 4D field theory using "vortex-like" configuration (via anomaly inflow). May obtain the 2D CFT for chiral Luttinger Liquid (discussed in the context of the fractional quantum Hall effect.) rather than free Fermion.

Discussion

- What actually is the theory on vortex-like configuration?
- Examples of spectral winding for the non-Abelian anomaly using anomaly inflow?
- Stability of spectral winding?
- Can we deduce stronger dynamical implications using spectral winding via the anomaly matching?