On large deviations of SLEs, real rational functions, and zeta-regularized determinants of Laplacians

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July 2023 @ OIST CFTPRGR Workshop

Joint work with Yilin Wang (IHES)

What is this talk all about?

1. *Schramm-Loewner evolution* (SLE_κ): random planar curves
2. Large deviations and *Loewner energy*: concentration phenomenon
3. Loewner energy / potential in terms of known quantities:
   (zeta-regularized) determinants of Laplace-Beltrami operators
4. Interpretation of minima?
   - semiclassical Virasoro conformal blocks in CFT
   - *Calogero-Moser systems* [Alberts, Byun, Kang, Makarov '22]

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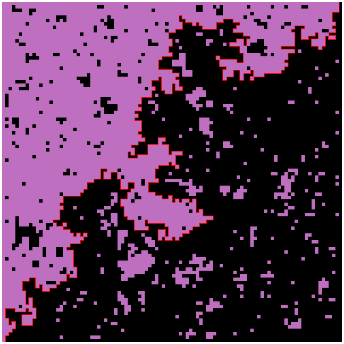
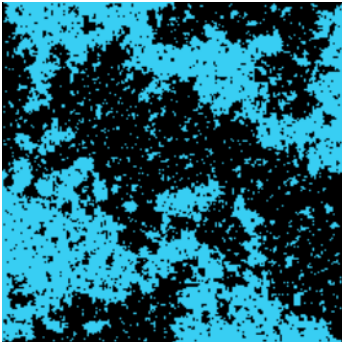
4. Interpretation of minima?
   - semiclassical Virasoro conformal blocks in CFT
   - *Calogero-Moser systems* [Alberts, Byun, Kang, Makarov '22]
5. Classification of minimizers? (not in this talk?)
   - real rational functions with prescribed critical points
   - *Shapiro-Shapiro conjecture* [B. & M. Shapiro '95]
6. Numerous further connections (not in this talk):
   - *Partition function of Coulomb gas* on Jordan loop
     [Johansson '21; Wiegmann, Zabrodin '21]
   - *Kähler potential* of WP metric on *univ. Teich. space* [Wang '19]
   - *Renormalized volume* in hyperbolic 3-space
     [Bridgeman, Bromberg, Vargas-Pallete, Wang '23+]
   - Connections to function theory... [Bishop '19]

What is SLE$_\kappa$?

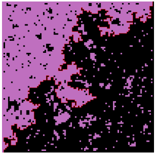


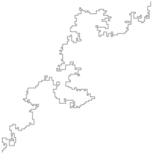


Universal 2D random path

Scaling limits of critical interfaces — SLE$_\kappa$ curves

- $\kappa > 0$ labels *universality class* (e.g. $\kappa = 3$ for Ising model)
- convergence weakly for probability measures on curves





$$\text{(critical) interface} \xrightarrow{\delta \to 0} \text{Schramm-Loewner evolution, } \mathrm{SLE}_\kappa$$

Usual proof strategy:

1. *tightness* (e.g. control via crossing estimates, RSW etc.)

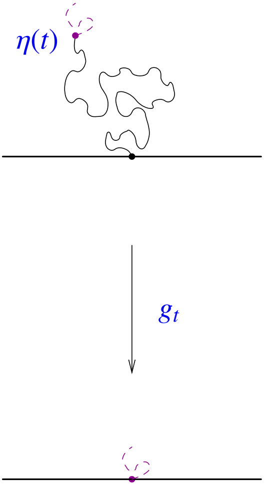
   [Aizenman & Burchard '99, Kemppainen & Smirnov '17, ...]

2. *identification* of the limit (e.g. via discrete holomorphic observable)

   [Kenyon '00, Chelkak & Smirnov '01–'11, ...] □

$\Longrightarrow$ **conformal invariance**

Loewner evolution of curves / slit domains



Thm. [Loewner '23]

Any **simple chordal curve** η

(more generally, a *locally growing* family of hulls)

can be encoded into a Loewner evolution

of conformal maps

$g_t : \mathbb{H} \setminus \eta[0,t] \to \mathbb{H}$ which solve the ODE

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W(t)}, \qquad g_0(z) = z,$$

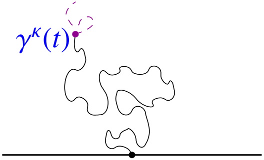
where W is a (continuous) real-valued function.

(Here, we have chosen the capacity parameterization.)

$W_t = g_t(\eta(t))$ on $\mathbb{R}$

↑ **Loewner driving function** $W \colon [0,\infty) \to \mathbb{R}$

Schramm-Loewner evolution, SLE$_\kappa$ (let's assume $\kappa < 8/3$)

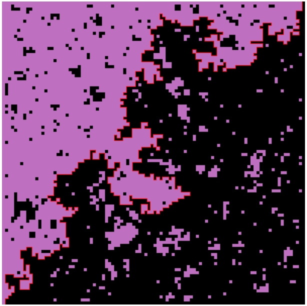


$g_t : \mathbb{H} \setminus \gamma^\kappa[0,t] \to \mathbb{H}$

$W_t = g_t(\gamma^\kappa(t)) = \sqrt{\kappa} B_t$

Thm. [Schramm '00]

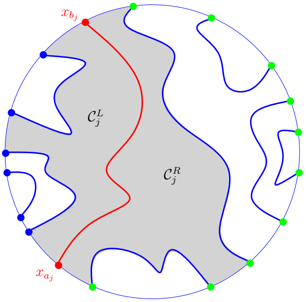
$\exists!$ one-parameter family $(\mathrm{SLE}_\kappa)_{\kappa \geq 0}$ of probability measures on chordal curves with **conformal invariance** and **domain Markov property**



Loewner driving process: **Brownian motion B of "speed" $\kappa \geq 0$**

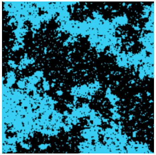
Multiple (chordal) SLE$_\kappa$ (let's assume $\kappa < 8/3$)

- family of **random chordal curves** $(\gamma_1^\kappa, \ldots, \gamma_N^\kappa)$ in $(D; x_1, \ldots, x_{2N})$
- connectivities encoded in **planar pairings** α of curve endpoints $\{\{x_{a_j}, x_{b_j}\}\}_{j=1,\ldots,N}$
- **re-sampling symmetry** ($\rightsquigarrow$ Markov chain)



Conditionally on $N-1$ of the curves, the remaining one is the chordal SLE$_\kappa$ in the random domain where it can live.

cf. many works: Cardy '03; Bauer, Bernard & Kytölä '05; Dubédat '06–'07; Kozdron & Lawler '07; Lawler '09; Kytölä & P. '16; Miller & Sheffield '16; P. & Wu '19; Miller, Sheffield & Werner '20; Beffara, P. & Wu '21, ...



MULTIPLE (CHORDAL) SLE$_\kappa$ (LET'S ASSUME $\kappa < 8/3$)

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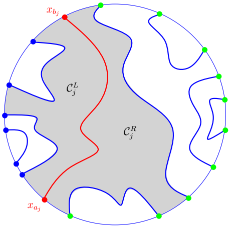

Thm. [Lawler, Schramm & Werner '03, ..., Beffara, P. & Wu '21]
For any fixed connectivity α of $2N$ points,
there exists a unique N-SLE$_\kappa$ probability measure $\mathbb{P}_\alpha^\#$.

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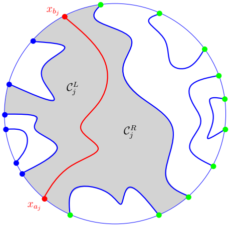

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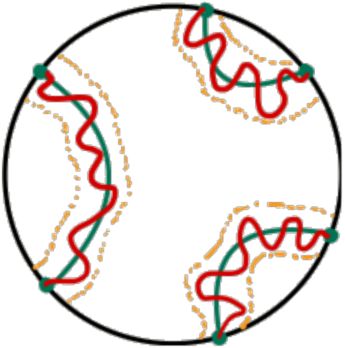
$$\frac{\mathrm{d}\mathbb{P}_\alpha}{\bigotimes_{1 \le i \le N} \mathrm{dP}_{\mathrm{SLE}}^{(i)}} := \exp\left(\frac{c(\kappa)}{2} m^{\mathrm{loop}}(D; \gamma_1^\kappa, \ldots, \gamma_N^\kappa)\right), \qquad \mathbb{P}_\alpha^\# = \frac{\mathbb{P}_\alpha}{|\mathbb{P}_\alpha|}$$

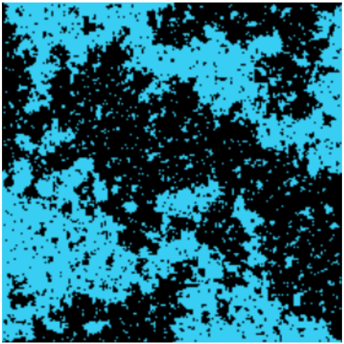
- m^{loop}: combinatorial expression involving *Brownian loop measure* μ_D^{loop}:

$$m^{\mathrm{loop}}(D; \gamma_1^\kappa, \ldots, \gamma_N^\kappa) = \int \max\Big(\#\{\text{chords } \gamma_j^\kappa \text{ hit by } \ell\} - 1,\ 0\Big)\, \mathrm{d}\mu_D^{\mathrm{loop}}(\ell)$$

- $c(\kappa) = \frac{(3\kappa-8)(6-\kappa)}{2\kappa} < 0$: parameter (central charge) depending on κ

Large deviations of SLE_κ as $\kappa \to 0+$

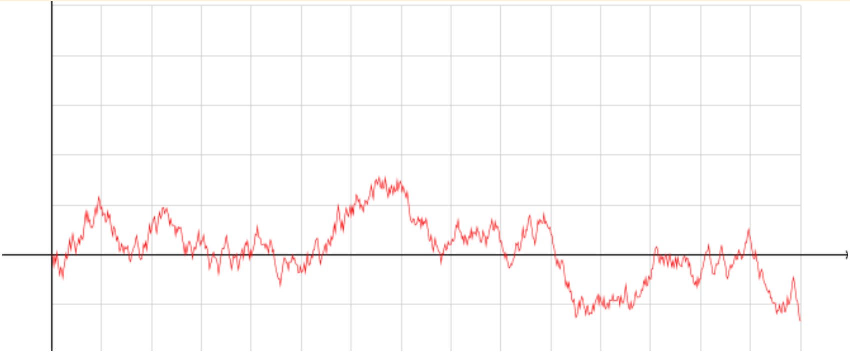




Large deviations for Brownian motion

- Let's consider **given continuous function** $W : [0, T] \to \mathbb{R}$ s.t. $W_0 = 0$. Idea:

" $\mathbb{P}[$ Brownian path $\sqrt{\kappa} B_{[0,T]}$ stays close to $W_{[0,T]}]$ $\overset{\kappa \to 0+}{\approx}$ $\exp\Big(- \frac{I_T(W)}{\kappa} \Big)$ "



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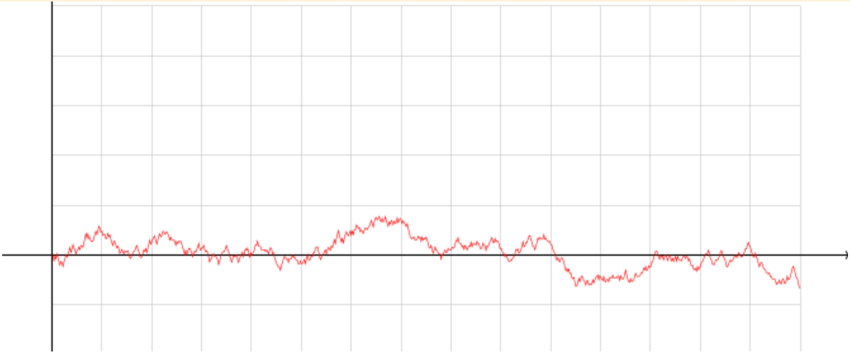

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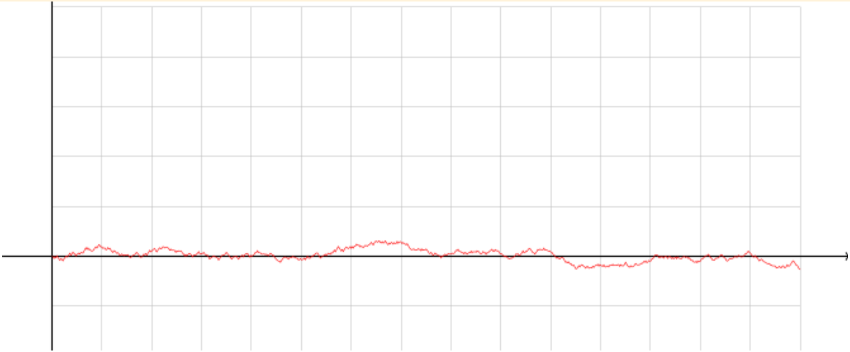

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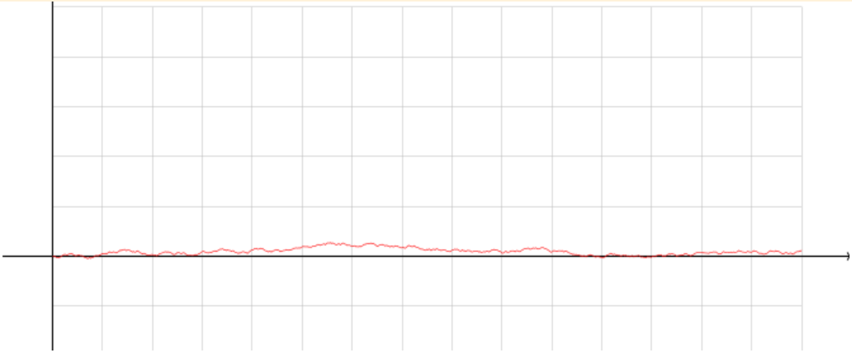

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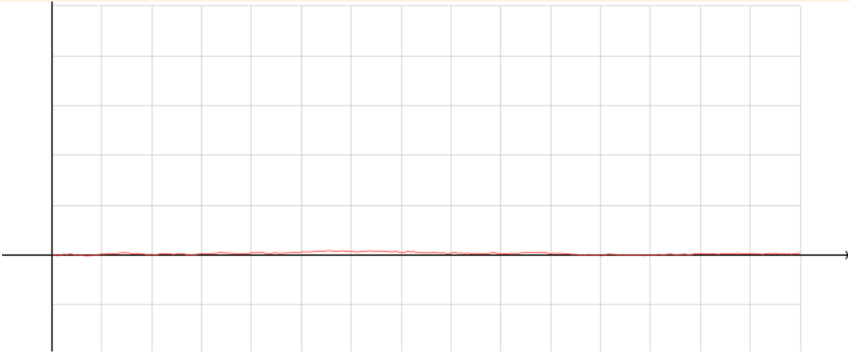

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Thm. [Schilder '66] (Large Deviation Principle for BM)

Fix $T > 0$. The random path $\sqrt{\kappa}B_{[0,T]}$ satisfies LDP in $C^0[0,T]$ with sup-norm, with good *rate function* $I_T(W) := \frac{1}{2}\int_0^T \Big(\frac{\mathrm{d}}{\mathrm{d}t} W_t\Big)^2 \mathrm{d}t$

$$\limsup_{\kappa \to 0+} \kappa \log \mathbb{P}[\sqrt{\kappa}B_{[0,T]} \in C] \leq - \inf_{W \in C} I_T(W) \quad \text{for any closed set } C$$

$$\liminf_{\kappa \to 0+} \kappa \log \mathbb{P}[\sqrt{\kappa}B_{[0,T]} \in O] \geq - \inf_{W \in O} I_T(W) \quad \text{for any open set } O$$

- finite time-window T
- $C^0[0,T] = \{W : [0,T] \to \mathbb{R} \text{ continuous}, \; W_0 = 0\}$
- topology: $\|W\|_\infty := \sup_{t \in [0,T]} |W_t|$

Large deviations of chordal SLE_κ as $\kappa \to 0+$

- Let's consider **given smooth curve** η in (D, x, y). Idea:

$$\text{“}\ \mathbb{P}[\mathrm{SLE}_\kappa \text{ curve stays close to } \eta] \overset{\kappa \to 0+}{\approx} \exp\Big(- \frac{I(\eta)}{\kappa} \Big) \text{”}$$

- **Decay rate: Loewner energy** of the curve η

defined as the *Dirichlet energy* of its driver W:

$$I(\eta) := \tfrac{1}{2} \int_0^\infty \Big(\tfrac{\mathrm{d}}{\mathrm{d}t} W_t\Big)^2 \mathrm{d}t \quad \in \quad [0, +\infty]$$

[Dubédat ’05; Friz & Shekhar ’17; Wang ’19; Bishop ’19, ...]

Thm. [Wang ’19; P. & Wang ’23]

The family of laws $(\mathbb{P}^\kappa)_{\kappa>0}$ of SLE_κ curves γ^κ satisfies LDP:

(for Hausdorff distance, with good rate function I)

$$\limsup_{\kappa \to 0+} \kappa \log \mathbb{P}^\kappa[\gamma^\kappa \in C] \leq - \inf_{\eta \in C} I(\eta) \qquad \text{for any closed set } C$$

$$\liminf_{\kappa \to 0+} \kappa \log \mathbb{P}^\kappa[\gamma^\kappa \in O] \geq - \inf_{\eta \in O} I(\eta) \qquad \text{for any open set } O$$

Large deviations of multichordal SLE_κ as $\kappa \to 0+$

- Let's consider **given smooth curves** $\bar{\eta} := (\eta_1, \ldots, \eta_N)$. Idea:

$$\text{“ } \mathbb{P}[\mathrm{SLE}_\kappa \text{ curves stay close to } \bar{\eta}] \overset{\kappa \to 0+}{\approx} \exp\Big(-\frac{I(\bar{\eta})}{\kappa}\Big) \text{ ”}$$

- Decay rate: $I(\bar{\eta}) \geq 0$, **Loewner energy** of the multichord $\bar{\eta}$

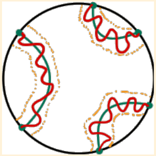
Thm. [P. & Wang '23]

The family of laws $(\mathbb{P}^\kappa)_{\kappa>0}$ of SLE_κ curves $\bar{\gamma}^\kappa$ satisfies LDP:

(for Hausdorff distance, with good rate function I)

$$\limsup_{\kappa \to 0+} \kappa \log \mathbb{P}^\kappa[\bar{\gamma}^\kappa \in C] \leq -\inf_{\bar{\eta} \in C} I(\bar{\eta}) \qquad \text{for any closed set } C$$

$$\liminf_{\kappa \to 0+} \kappa \log \mathbb{P}^\kappa[\bar{\gamma}^\kappa \in O] \geq -\inf_{\bar{\eta} \in O} I(\bar{\eta}) \qquad \text{for any open set } O$$



Proof idea: Schilder thm for BM, Varadhan's lemma + *careful analysis*

Intrinsic object: Loewner potential

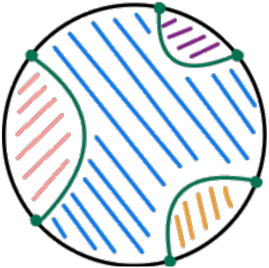
- **Multi-chord Loewner energy** of curves $\bar{\eta} := (\eta_1, \ldots, \eta_N)$:

$$I_D(\bar{\eta}) := 12 \left(\mathcal{H}_D(\bar{\eta}) - \inf_{\bar{\gamma}} \mathcal{H}_D(\bar{\gamma}) \right)$$

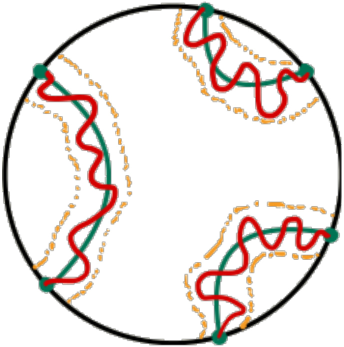
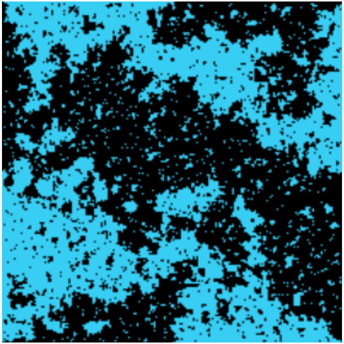
- **Loewner potential** $\mathcal{H}_D(\bar{\eta})$ of curves $\bar{\eta} := (\eta_1, \ldots, \eta_N)$:

$$\mathcal{H}_D(\bar{\eta}) := \frac{1}{12} \sum_{j=1}^{N} I_D(\eta_j) + m_D^{\text{loop}}(\bar{\eta}) - \frac{1}{4} \sum_{j=1}^{N} \log P_D(x_{a_j}, x_{b_j})$$

- $I_D(\eta) := \frac{1}{2} \int_0^\infty (\frac{\mathrm{d}}{\mathrm{d}t} W_t)^2 \mathrm{d}t$
  one-curve Loewner energy
- "interaction": $m_D^{\text{loop}}(\bar{\eta})$
  Brownian loop measure term
- $P_D(x_{a_j}, x_{b_j})$ boundary Poisson kernel
- x_{a_j}, x_{b_j} endpoints of curve η_j



Loewner potential

IN ANOTHER FORM

Loewner potential – more intuitive formula

As $\mathcal{H}(\bar{\eta})$ is a bit complicated, let's write it differently:

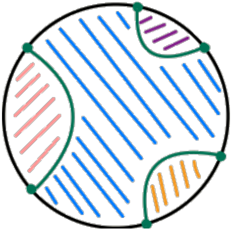
Thm. [P. & Wang '23]

For any smooth $\bar{\eta}$ in bounded smooth domain $(D; x_1, \ldots, x_{2N})$,

$$\mathcal{H}_D(\bar{\eta}) = \log \det{}_\zeta \Delta_D - \sum_{\text{c.c. } C} \log \det{}_\zeta \Delta_C - \frac{N}{2} \log \pi$$

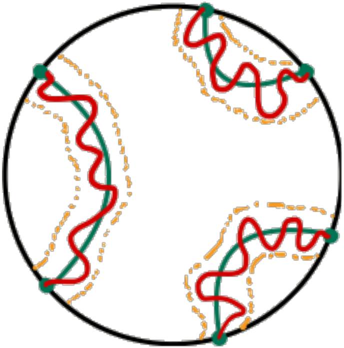
Proof idea: Both sides have the same conformal covariance; use *Polyakov-Alvarez* anomaly formula (for domains with corners) [Aldana, Kirsten, Rowlett '20] □

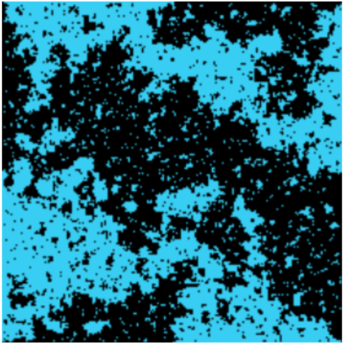
- $\log \det_\zeta \Delta$ zeta-regularized determinant of Laplacian Δ with Dirichlet b.c.
- sum over *connected components* C of $D \setminus \bigcup_i \eta_i$
- $\frac{1}{2} \log \pi \approx 0.5724$ *universal constant*
- motivated by loop case & rel. to geometry: [Wang '19]



NB: Also makes sense on Riemannian surfaces (depends on metric).

Potential/energy minima

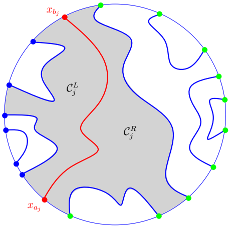




Conformal blocks in CFT ?

Recall: multiple (chordal) SLE_κ (let's assume $\kappa < 8/3$)

- family of **random chordal curves** $(\gamma_1^\kappa, \ldots, \gamma_N^\kappa)$ in $(D; x_1, \ldots, x_{2N})$
- connectivities encoded in **planar pairings** α of curve endpoints $\{\{x_{a_j}, x_{b_j}\}\}_{j=1,\ldots,N}$
- **re-sampling symmetry** ($\rightsquigarrow$ Markov chain)



Thm. [Lawler, Schramm & Werner '03, ..., Beffara, P. & Wu '21]
For any fixed connectivity α of $2N$ points,
there exists a unique N-SLE_κ probability measure $\mathbb{P}_\alpha^\#$.

- describe interaction of curves by "(pure) partition function" (total mass)

$$\mathcal{Z}_\alpha(D; x_1, \ldots, x_{2N}) := |\mathbb{P}_\alpha|(D; x_1, \ldots, x_{2N}) \prod_{j=1}^{N} P_D(x_{a_j}, x_{b_j})^{\frac{6-\kappa}{2\kappa}}$$

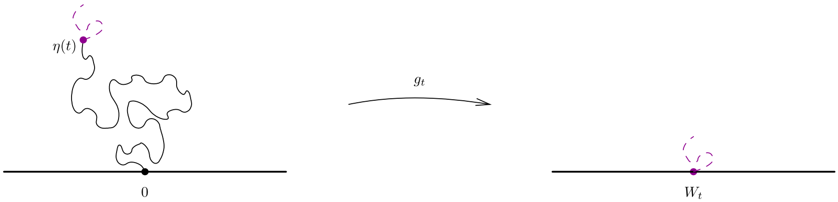
- Loewner driving process in $D = \mathbb{H}$ for curve γ_1^κ:

$$\mathrm{d}W_t = \sqrt{\kappa}\,\mathrm{d}B_t + \kappa\,\partial_1 \log \mathcal{Z}_\alpha(W_t, g_t(x_2), g_t(x_3), \ldots, g_t(x_{2N}))\,\mathrm{d}t$$

- CFT: [Cardy '84; Bauer-Bernard '02] "insert" fields $\Phi_{1,2}(x_j) \Longrightarrow$ BPZ equations

"SLE(κ) field $\Phi_{1,2}$" of weight $h_{1,2} = \frac{6-\kappa}{2\kappa}$

"insert" $\Phi_{1,2}$ at points $x_1 < x_2 < \cdots < x_{2N}$ [Cardy '84; Bauer-Bernard '02]



$$\mathrm{d}W_t = \sqrt{\kappa}\,\mathrm{d}B_t + \kappa\,\partial_1 \log \mathcal{Z}_\alpha(W_t, g_t(x_2), g_t(x_3), \ldots, g_t(x_{2N}))\,\mathrm{d}t$$

- parameter $\kappa > 0$, central charge $c = \frac{1}{2\kappa}(3\kappa - 8)(6-\kappa) = 13 - 6\left(\frac{\kappa}{4} + \frac{4}{\kappa}\right)$
- singular vector $\left(L_{-2} - \frac{3}{2(2h_{1,2}+1)} L_{-1}^2\right) v_{1,2}$
- (together with translation invariance) gives rise to **PDE system** $\forall\, i$

$$\left\{ \frac{\kappa}{2} \frac{\partial^2}{\partial x_i^2} + \sum_{j=1}^{n} \left(\frac{2}{x_j - x_i} \frac{\partial}{\partial x_j} - \frac{2h_{1,2}(\kappa)}{(x_j - x_i)^2} \right) \right\} \underbrace{\langle \Phi_{1,2}(x_1) \cdots \Phi_{1,2}(x_{2N}) \rangle}_{\mathcal{Z}(x_1, x_2, \ldots, x_{2N})} = 0$$

Minima $\implies$ Semiclassical Virasoro conformal blocks

- Fix domain data $D = \mathbb{H}$ and $x_1 < \cdots < x_{2N}$ and connectivity α
- Set $\mathcal{U}(x_1, \ldots, x_{2N}) := 12 \inf_{\bar{\gamma}} \mathcal{H}_{\mathbb{H}; x_1, \ldots, x_{2N}}(\bar{\gamma})$ (minimum potential)

Thm. [P. & Wang '23]

$$\frac{1}{2}(\partial_j \mathcal{U}(x_1, \ldots, x_{2N}))^2 - \sum_{i \neq j} \frac{2}{x_i - x_j} \partial_i \mathcal{U}(x_1, \ldots, x_{2N}) = \sum_{i \neq j} \frac{6}{(x_i - x_j)^2} \quad \forall j$$

Proof: Study $\mathcal{U}$ & use self-similarity of Loewner flow of geodesic multichords □

- "Semiclassical limit" of Belavin-Polyakov-Zamolodchikov PDEs in conformal field theory (on $\hat{\mathbb{C}}$, from Virasoro symmetry)
- Appears also in the physics literature, e.g. [Teschner '11] and [Litvinov, Lukyanov, Nekrasov, Zamolodchikov '14]

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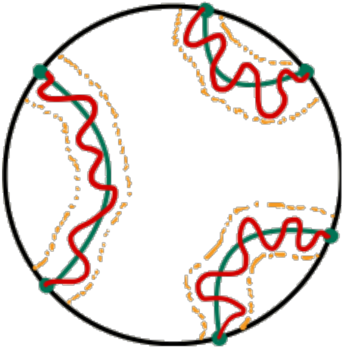
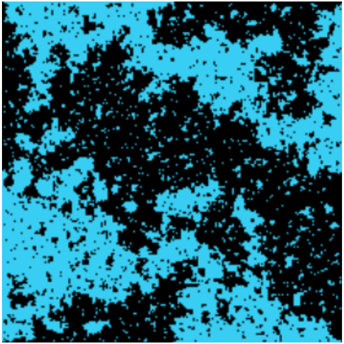
- Rigorously: SLE partition functions $\mathcal{Z}^\kappa$, s.t. $-\kappa \log \mathcal{Z}^\kappa \xrightarrow{\kappa \to 0} \mathcal{U}$
- [Litvinov, Lukyanov, Nekrasov, Zamolodchikov '14] also point out relation to Painlevé VI and AGT correspondence

Potential minimizers

⟹ optimal curves





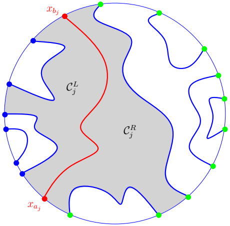
Shapiro conjecture (special case)

Potential minimizers $\Longrightarrow$ Geodesic multichords

Easy observation. SLE$_\kappa$ with $\kappa = 0$ is just the *hyperbolic geodesic*.

Lemma. Any minimizer of $\mathcal{H}(\bar{\eta})$ is a *geodesic multichord*.

$\bar{\eta} := (\eta_1, \ldots, \eta_N)$ is a **geodesic multichord** if **for each** $j \in \{1, 2, \ldots, N\}$**, the chord** η_j **is *hyperbolic geodesic* in its own component.**



Question: How many minimizers are there?

Key: Classify geodesic multichords!

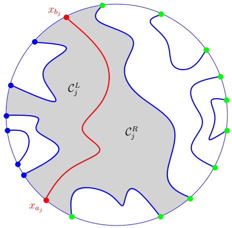
Potential minimizers $\Longrightarrow$ Geodesic multichords

Easy observation. SLE$_\kappa$ with $\kappa = 0$ is just the *hyperbolic geodesic*.

Lemma. $\bar{\eta} \mapsto \mathcal{H}(\bar{\eta})$ is lower semicontinuous (for Hausdorff metric) and has compact sublevel sets. In particular, minimizers of $\mathcal{H}(\bar{\eta})$ exist.

Lemma. Any minimizer of $\mathcal{H}(\bar{\eta})$ is a *geodesic multichord.*

$\bar{\eta} := (\eta_1, \ldots, \eta_N)$ is a **geodesic multichord** if **for each $j \in \{1, 2, \ldots, N\}$, the chord η_j is *hyperbolic geodesic* in its own component.**



Question: How many minimizers are there?

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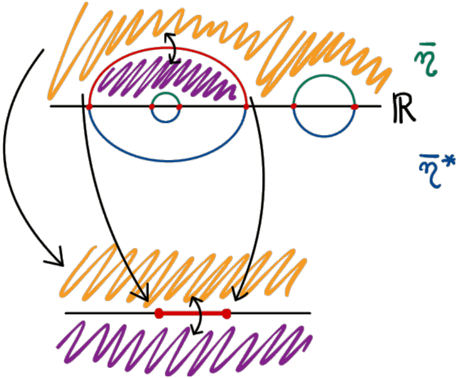
Geodesic multichords $\Longrightarrow$ Real rational functions

Lemma. Any minimizer of $\mathcal{H}(\bar{\eta})$ is a geodesic multichord*.

Proposition. Let $\bar{\eta}$ be a geodesic multichord in $\mathbb{H}$. The union of $\bar{\eta}$, its complex conjugate $\bar{\eta}^*$, and the real line is the real locus of a rational function of degree $N+1$ with critical points $\{x_1, \ldots, x_{2N}\}$.



$\forall\, j$, η_j is *hyperbolic geodesic* in its own component

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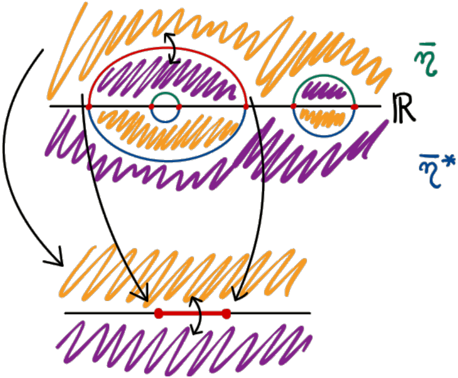


* $\forall\, j$, η_j is *hyperbolic geodesic* in its own component

Potential minimizers ⟹ Shapiro conjecture

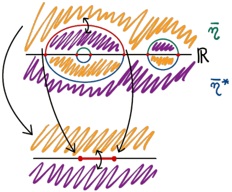
Thm. [P. & Wang '23]

- Each *minimizer* gives rise to unique* *rational function* on $\mathbb{C} \cup \{\infty\}$ of degree $N+1$ with $2N$ critical points on $\mathbb{R}$.
- $\exists!$ potential minimizer for each connectivity α.
- In particular, $\exists$ exactly* $\frac{1}{N+1}\binom{2N}{N}$ *rational functions* of deg. $N+1$ with given $2N$ critical points on $\mathbb{R}$.

$\star$ (up to post-composition by Möbius map)

Proof: Explicit construction & upper bound result [Goldberg '91] □

Cor. (Shapiro conjecture)
If all critical points of rational function are **real**, then it's a **real rational function***.



- special case of Shapiro conjecture [B. & M. Shapiro '95]
- first proven: [Eremenko & Gabrielov '00]
- general case: [Mukhin, Tarasov & Varchenko '09; Levinson & Purbhoo '21]


Thanks!