Hamiltonian Paths, Liouville Quantum Gravity and KPZ

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*arXiv:2305.02188**

New trends of conformal theory from probability to gravity

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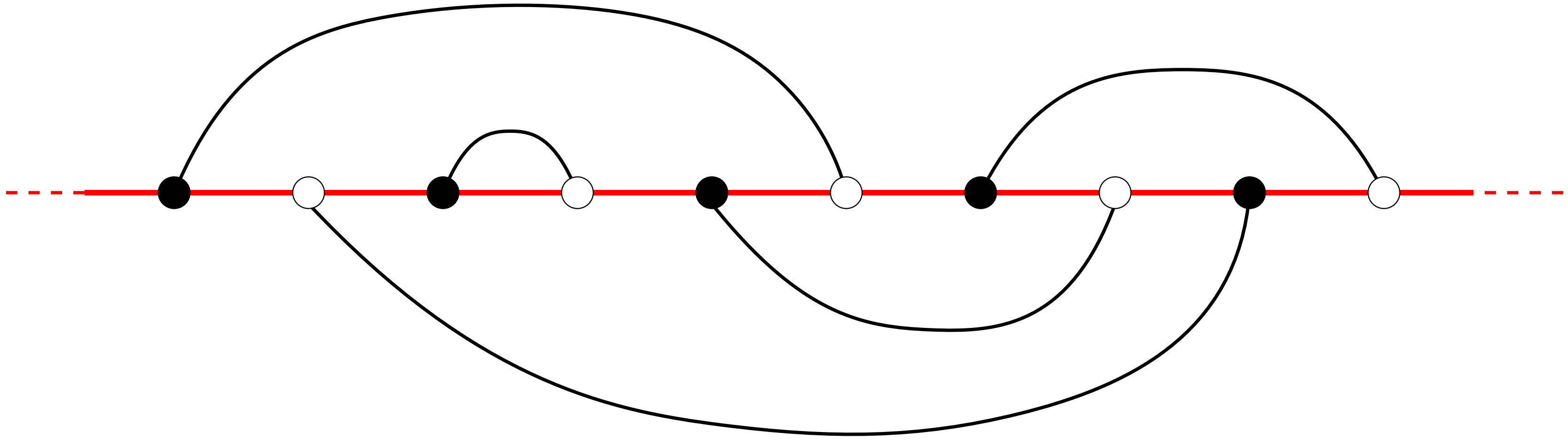
1. An enumeration problem that still holds out against combinatorialists (*)



- take an <u>infinite line</u> in the plane carrying a sequence of $2N$ <u>alternating</u> black and white points,
- connect all black points to white points by N <u>non-crossing</u> arches drawn above and/or below the line,
- call z_N the number of different ways to do so. Formula for z_N ?

(*) introduced in E.Guitter, C. Kristjansen, J. Nielsen 1999

```
0 1
1 2
2 8
3 40
4 228
5 1424
6 9520
7 67064
8 492292
9 3735112
10 29114128
11 232077344
12 1885195276
13 15562235264
14 130263211680
15 1103650297320
16 9450760284100
17 81696139565864
18 712188311673280
19 6255662512111248
20 55324571848957688
21 492328039660580784
22 4406003100524940624
23 39635193868649858744
24 358245485706959890508
25 3252243000921333423544
26 29644552626822516031040
27 271230872346635464906816
28 2490299924154166673782584
29 22939294579586403144527440
30 211949268051816569236796848
31 1963919128426791258770276024
32 18246482008315207478524287044
33 169953210523325203868381657400
34 1586759491069775179474823509344
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We expect $z_N \underset{N\to\infty}{\sim} \varkappa \dfrac{\mu^{2N}}{N^{2-\gamma}}$. Values of $\mu, \varkappa, \gamma$?

From the exact enumeration data, we may extract

$$\mu^2 = 10.113 \pm 0.001$$

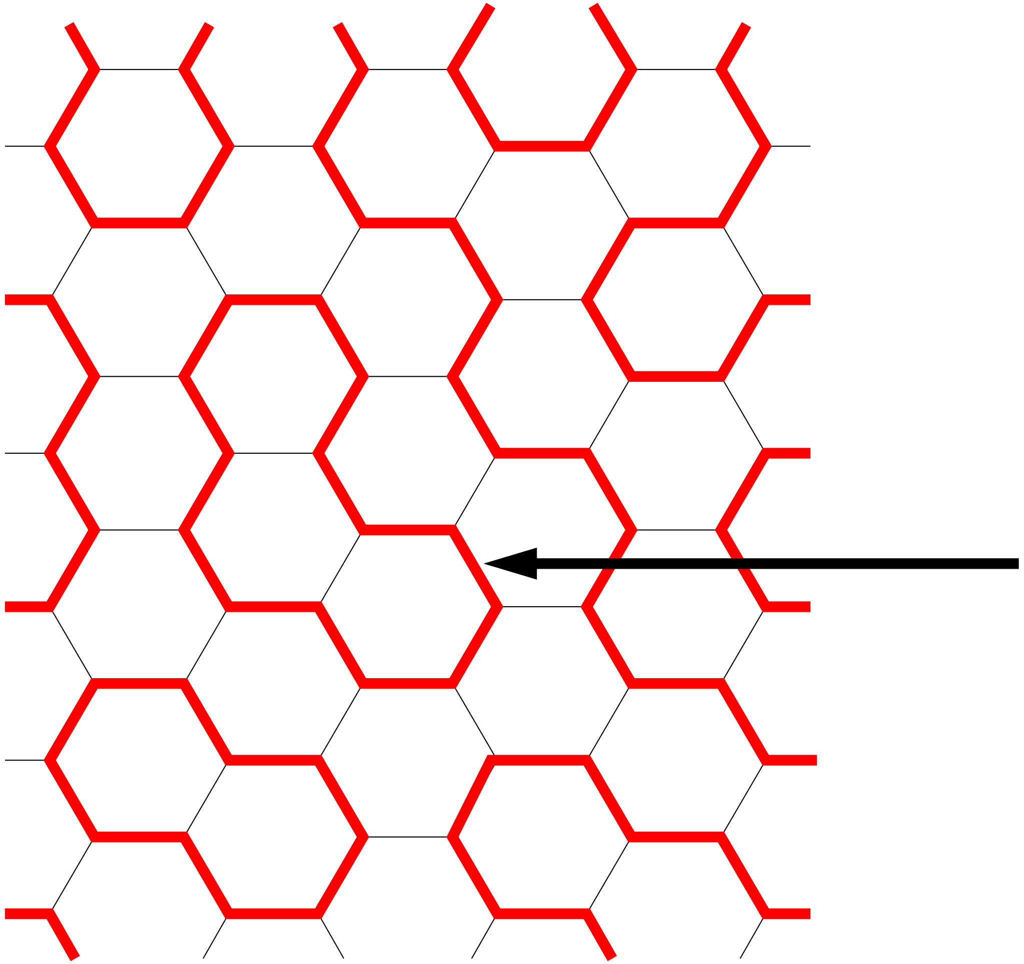
$$\gamma = -0.77 \pm 0.01$$

<u>Conjecture</u> (E. Guitter, C. Kristjansen, J. Nielsen 1999)

$$\gamma = -\frac{1+\sqrt{13}}{6} = -0.76759\cdots$$

2. Where bees come to the rescue

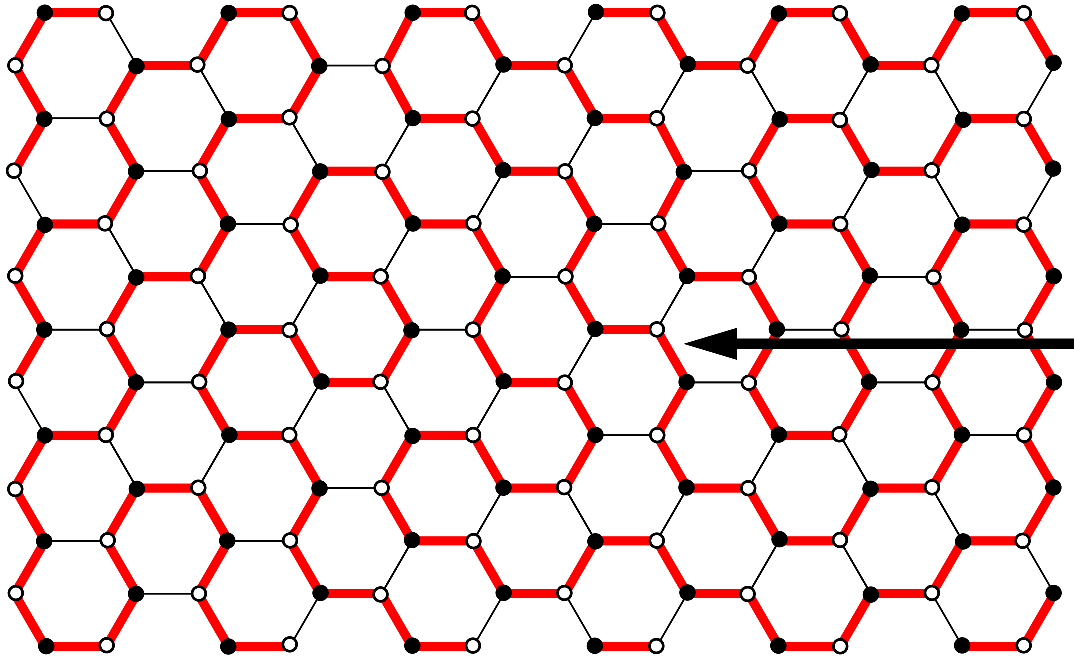
Statistical model on the honeycomb lattice



FPL(n) model on the honeycomb lattice

Fully Packed Loops := Loops drawn on the edges of the honeycomb lattice, and which visit all the vertices of the lattice

Assign a weight n to each loop



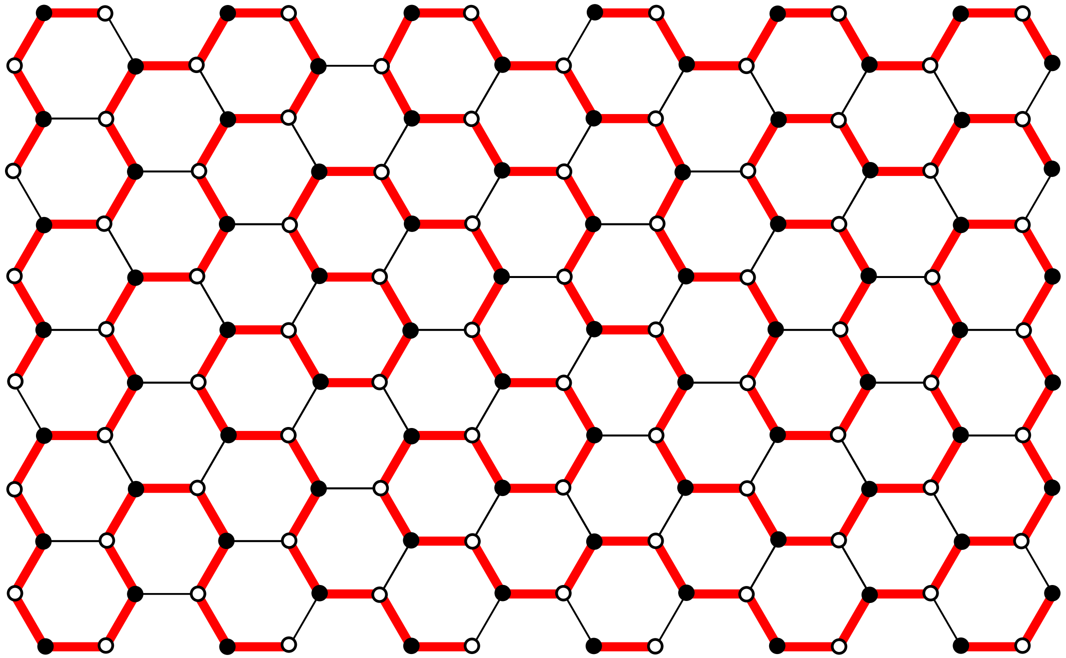
FPL(n) model on the honeycomb lattice

Honeycomb lattice
= the regular bicubic lattice

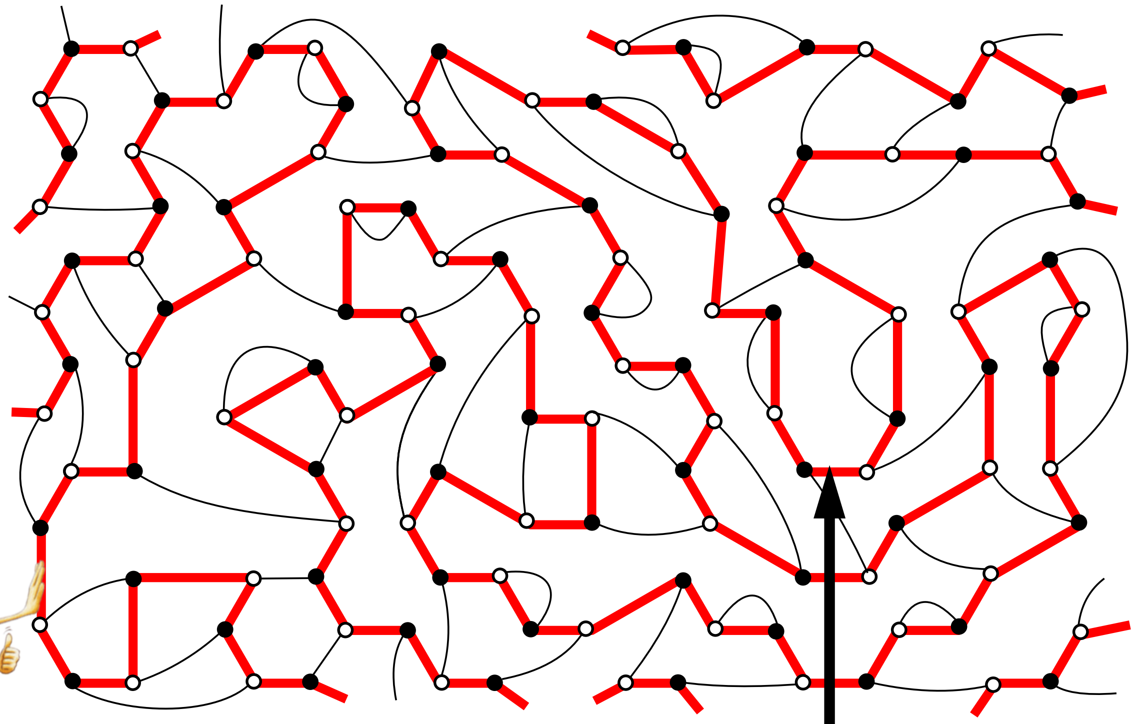
bicolored in black and white

all vertices have degree 3

Before « wine and cheese »



After « wine and cheese »



Honeycomb lattice:
the regular bicubic lattice

bicolored in black and white

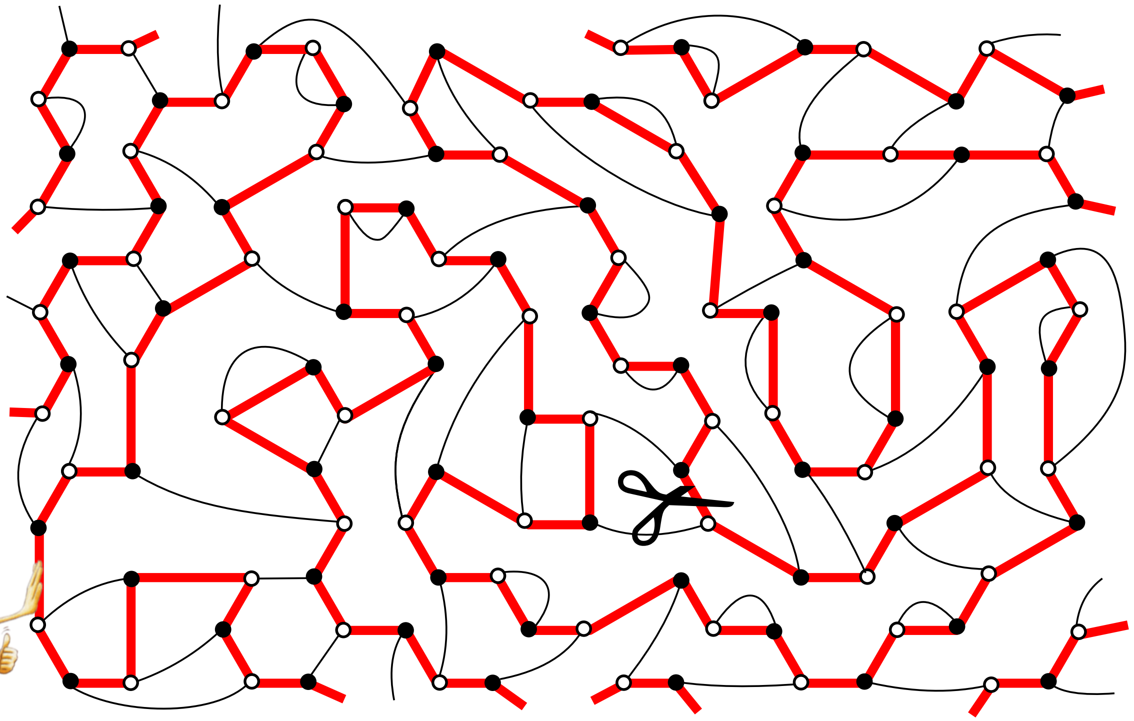
all vertices have degree 3

Random version:
random bicubic planar map

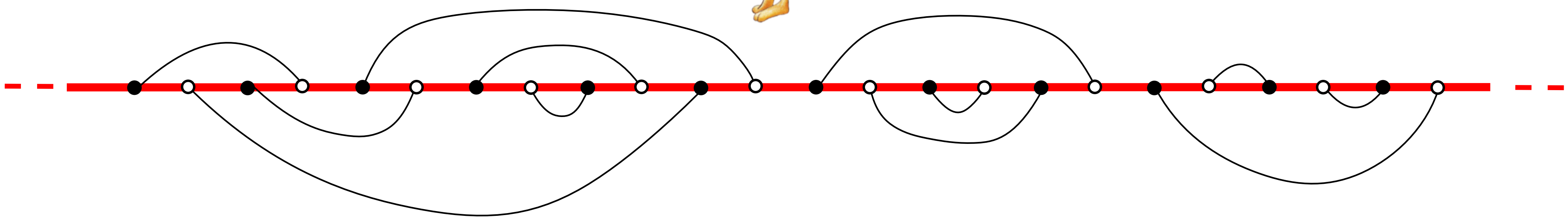
`Gravitational version'

= FPL(n) model on a random bicubic planar map

Taking the $n \to 0$ limit corresponds to selecting configurations with a single loop visiting all the vertices of the map



Cut the loop at some edge and stretch it into a straight line



Our combinatorial problem is nothing but the problem of a Hamiltonian cycle on a random bicubic map

V. Knizhnik, A. Polyakov, A. Zamolodchikov & DDK 1988
D. & Sheffield 2011

3. The KPZ relations

<u>Regular lattice</u>

Critical system described by a Conformal Field Theory with central charge c

Correlation function of operators $\Phi_{h_i,c}$ with conformal weight h_i

$$\langle \bar{\Phi}_{h_i,c}(0)\Phi_{h_i,c}(r)\rangle \sim \text{const.}\ \frac{1}{r^{4h_i}}$$

<u>Random planar map of fixed area A</u>

Partition function $\mathcal{Z}_A \sim \text{const.}\ \mu^A\, A^{\gamma(c)-3}$

$$\gamma(c) = \frac{1}{12}\left(c - 1 - \sqrt{(1-c)(25-c)}\right)$$

(Unnormalized) correlator

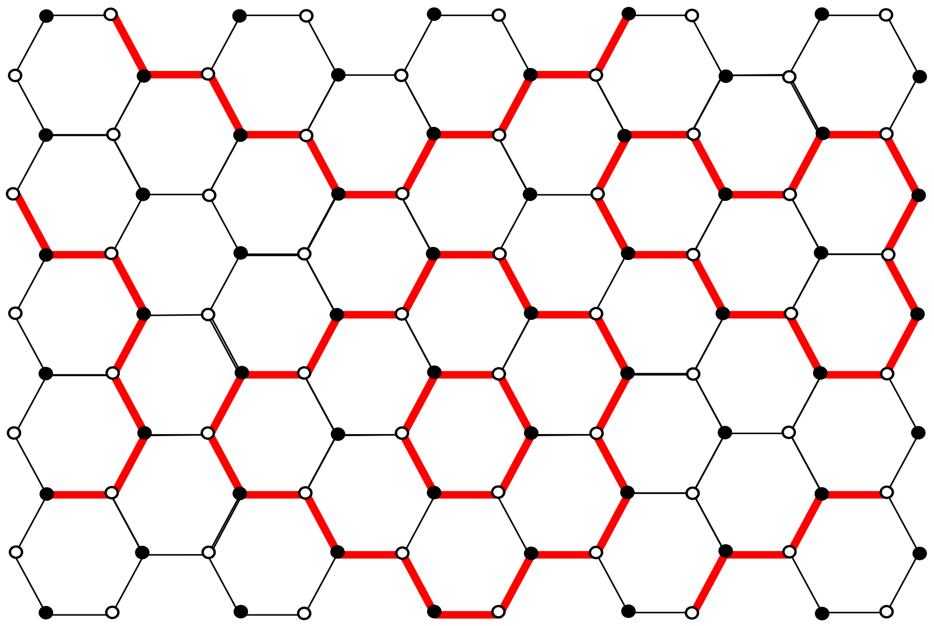
$$\mathcal{Z}_A \,\langle \prod_i \Phi_{h_i,c}\rangle_A \sim \text{const.}\ \mu^A\, A^{\sum_i\{1-\Delta(h_i,c)\}+\gamma(c)-3}$$

$$\Delta(h,c) = \frac{\sqrt{1-c+24\,h} - \sqrt{1-c}}{\sqrt{25-c} - \sqrt{1-c}}$$

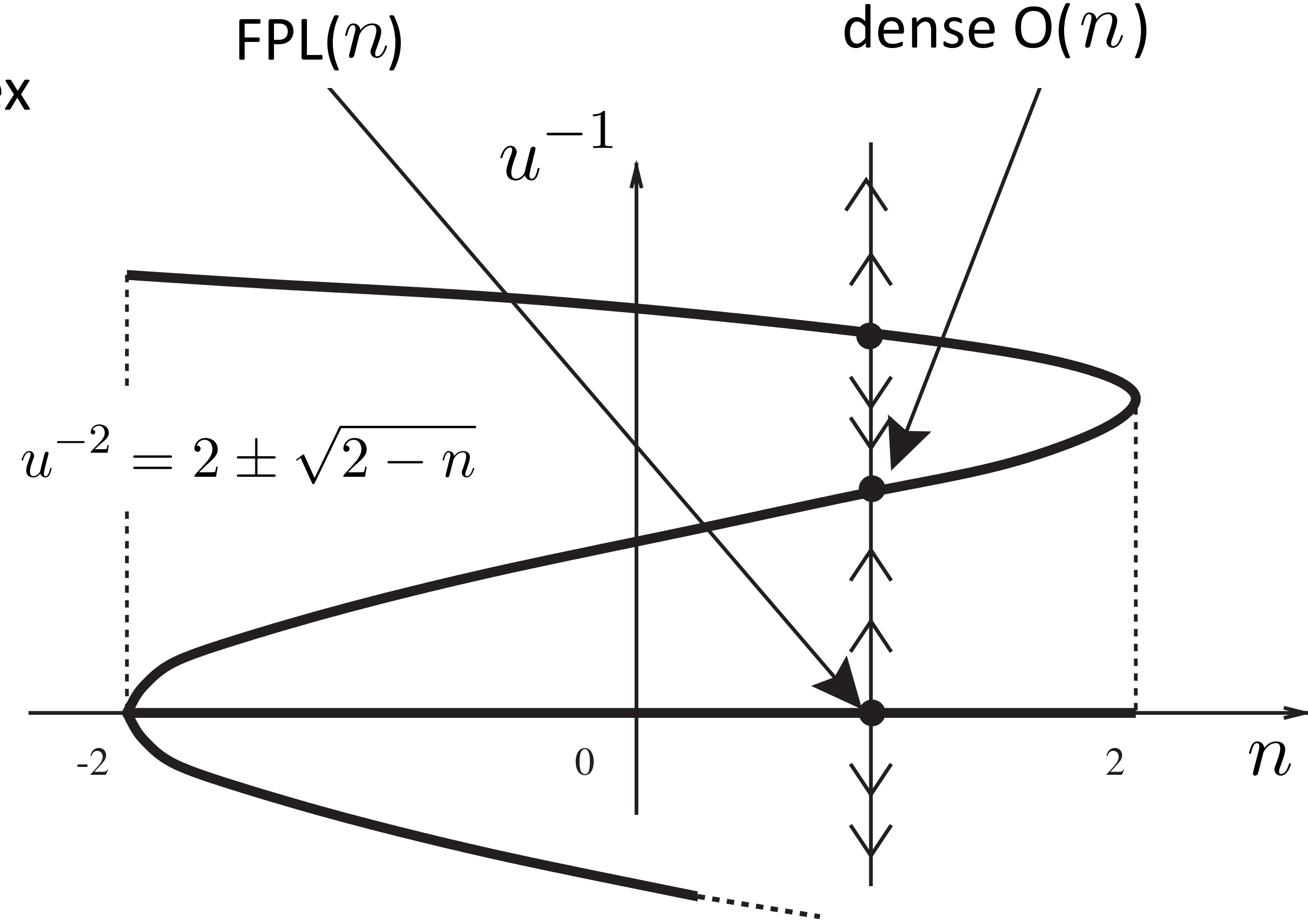
4. The FPL model on the honeycomb lattice

N. Reshetikhin 1991 / H. Blöte and B. Nienhuis 1994 / M.Batchelor, J. Suzuki and C. Yung 1994 / J. Kondev, J. de Gier, B. Nienhuis 1996 / J. Jacobsen, J. Kondev 1998 / T. Dupic, B. Estienne and Y. Ikhlef 2016, 2019

O(n) loop model: weight u per visited vertex



FPL(n) obtained by taking $u \to \infty$

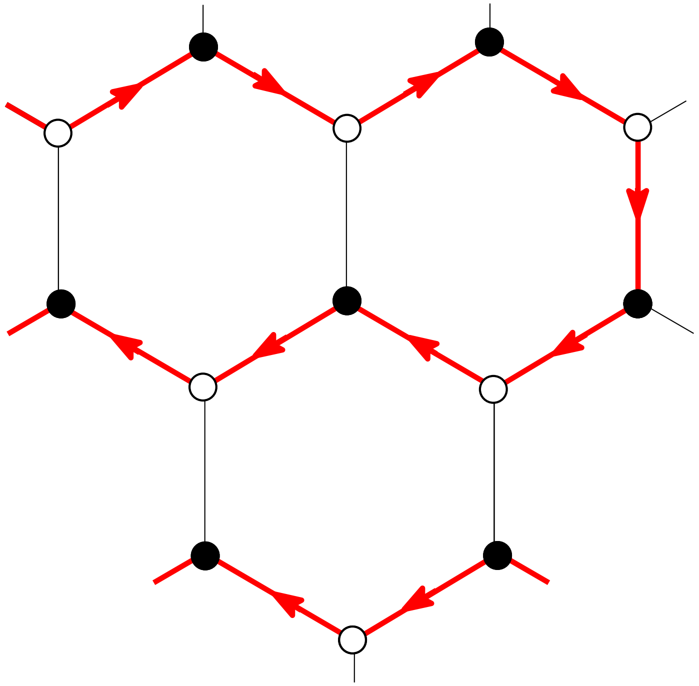


$$c_{\text{FPL}}(n) = c_{\text{dense}}(n) + 1$$

H. Blöte and B. Nienhuis 1994

→ value at $n = 2$

Why $c_{\text{FPL}}(2) = 2$ whereas $c_{\text{dense}}(2) = 1$?



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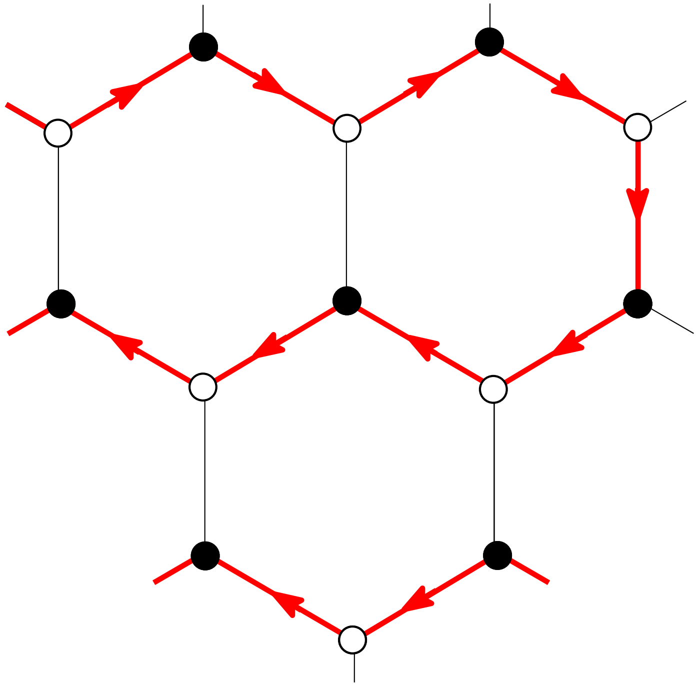

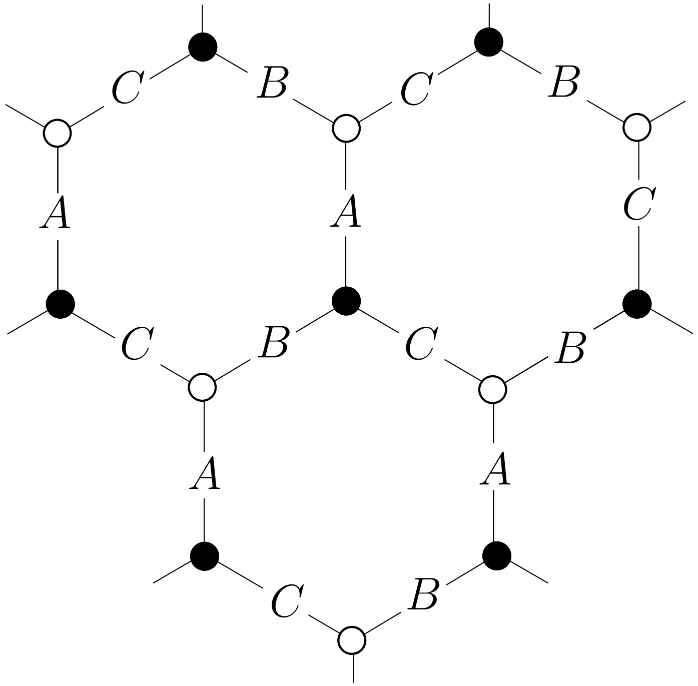

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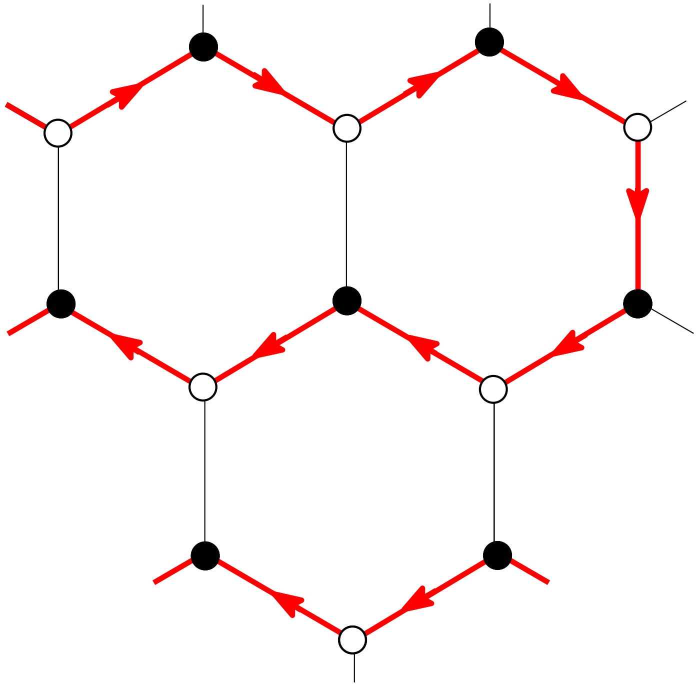

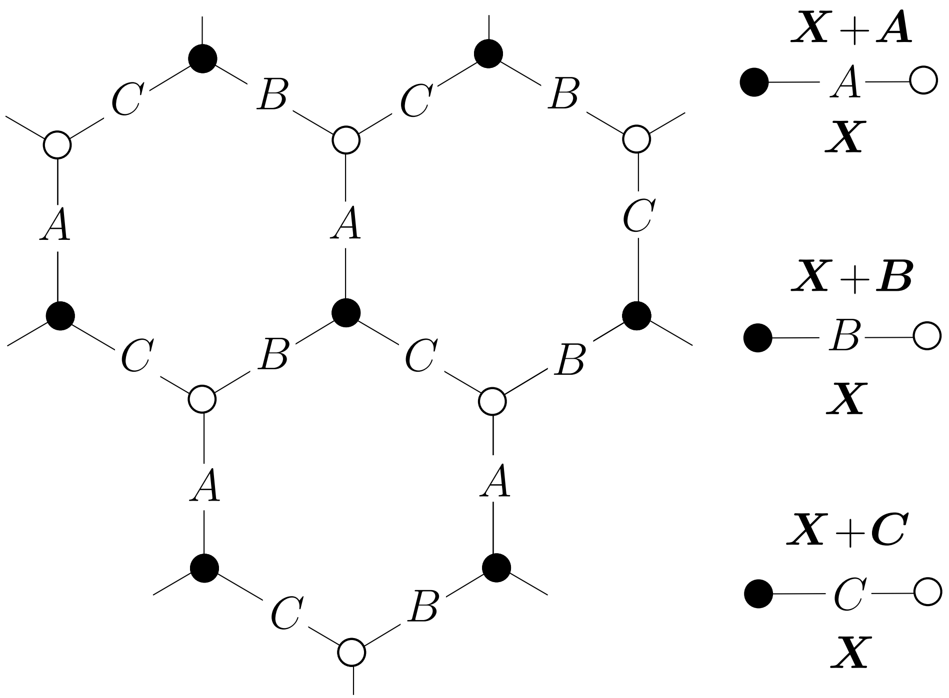

$c_{\mathrm{FPL}}(n) = c_{\mathrm{dense}}(n) + 1$ H. Blöte and B. Nienhuis 1994

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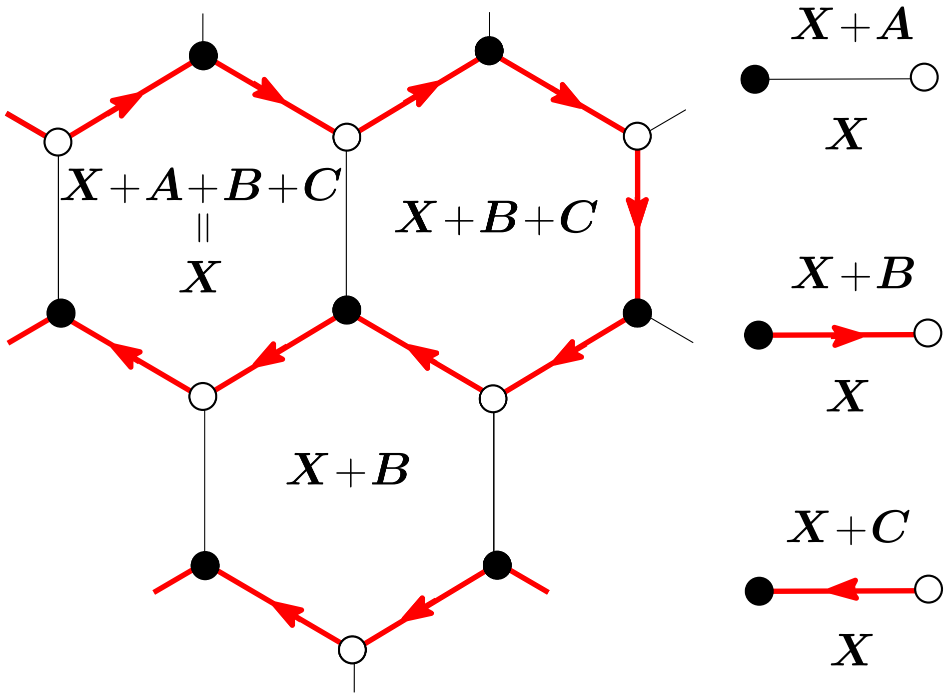

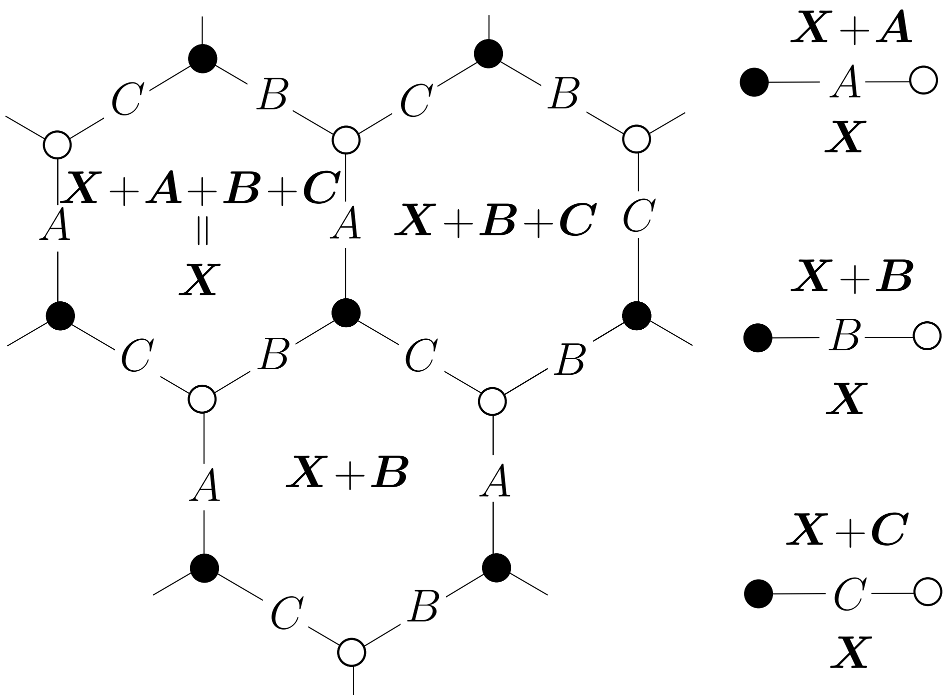

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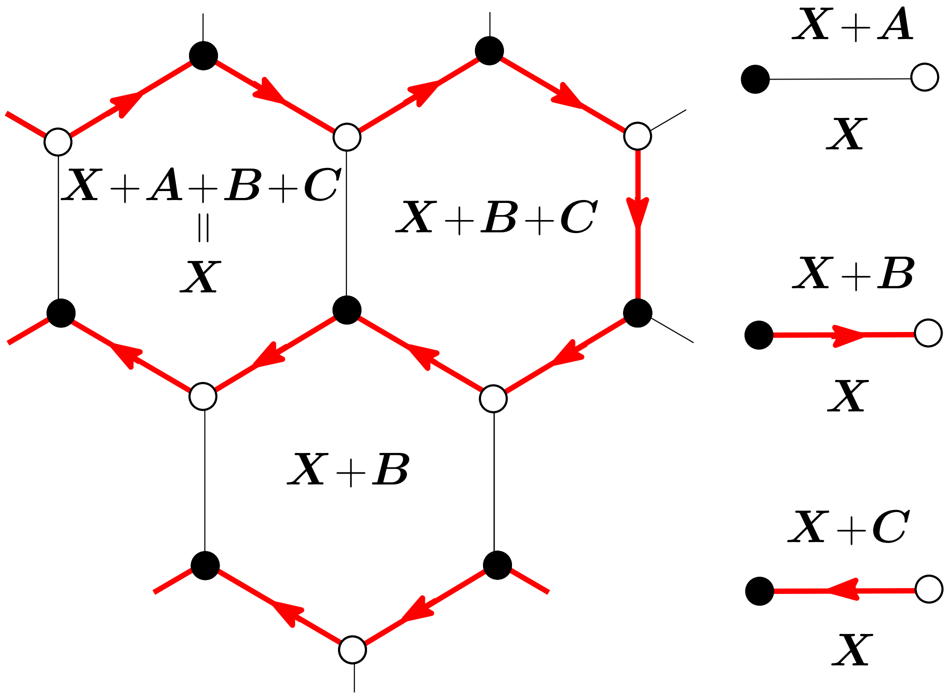
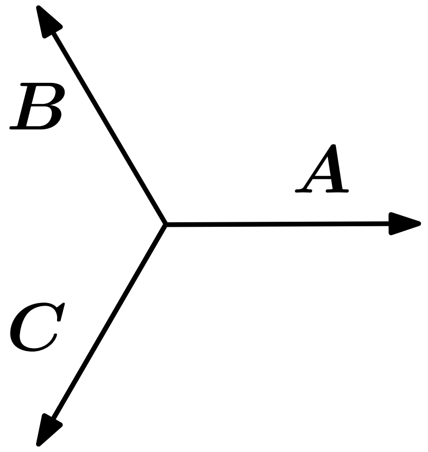


$$\boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C} = \boldsymbol{0}$$



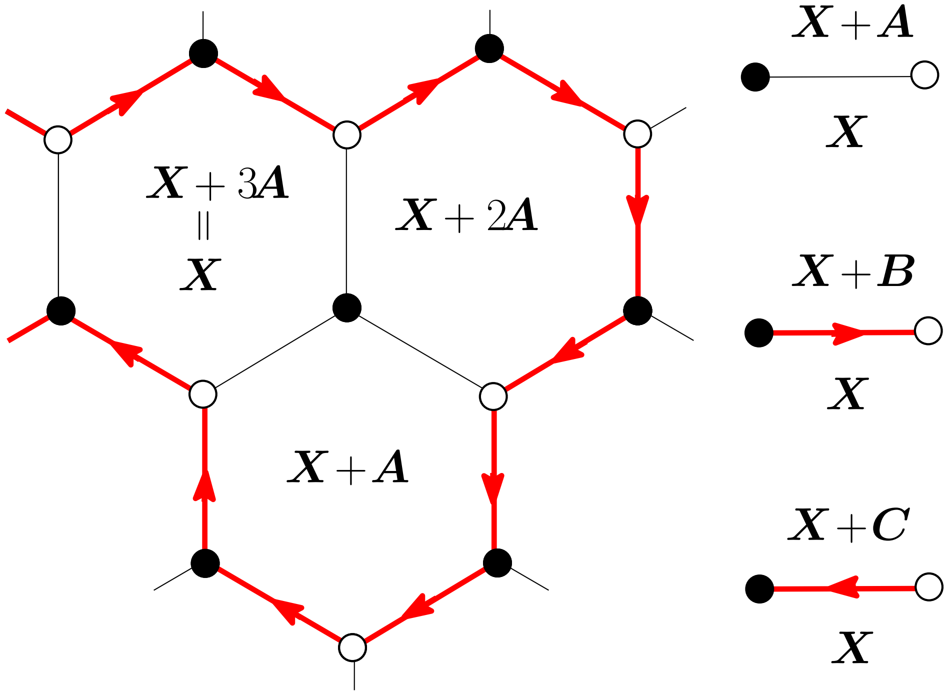
$\boldsymbol{X}$ 2-component « height » variable

$$c_{\mathrm{FPL}}(n) = c_{\mathrm{dense}}(n) + 1$$

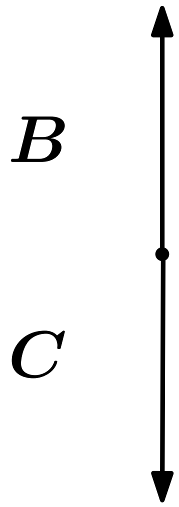
H. Blöte and B. Nienhuis 1994

→ value at $n = 2$

Why $c_{\mathrm{FPL}}(2) = 2$ whereas $c_{\mathrm{dense}}(2) = 1$?



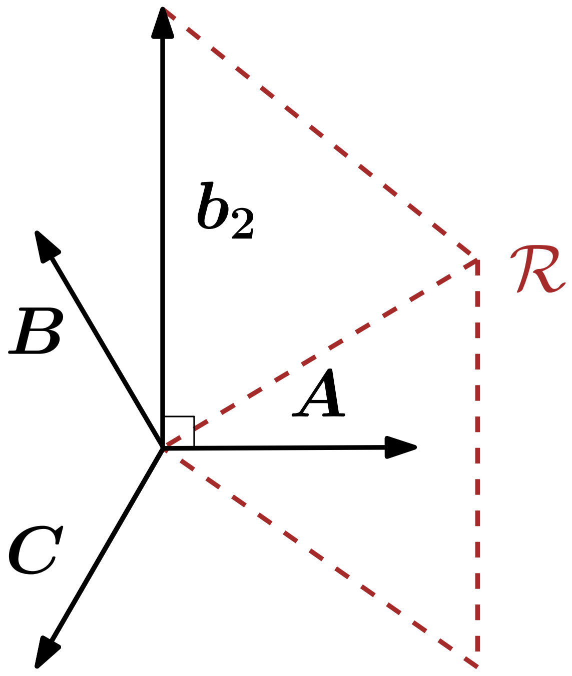
$$\boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C} = \mathbf{0}$$ and $$\boldsymbol{A} = \mathbf{0}$$



$\boldsymbol{X}$ 1 component « height » variable

Effective Coulomb Gas description of FPL on honeycomb

J. Kondev, J. de Gier, B. Nienhuis 1996



$$\boldsymbol{A} := \left(\frac{1}{\sqrt{3}}, 0\right) \;, \quad \boldsymbol{B} := \left(-\frac{1}{2\sqrt{3}}, \frac{1}{2}\right) \;, \quad \boldsymbol{C} := \left(-\frac{1}{2\sqrt{3}}, -\frac{1}{2}\right)$$

$$\boldsymbol{b_2} := \boldsymbol{B} - \boldsymbol{C} = (0,1)$$

Coarse grained variable $\boldsymbol{\Psi}(x) = \langle \boldsymbol{X} \rangle$ at position x

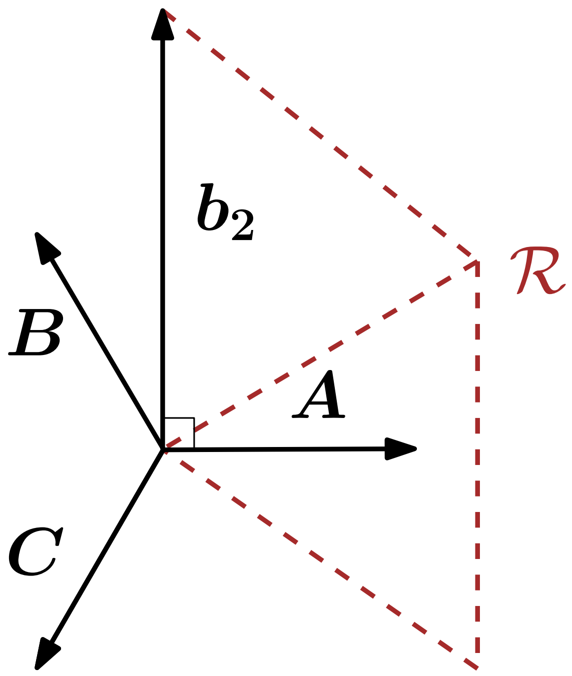
$$\boldsymbol{\Psi} = \psi_1 \boldsymbol{A} + \psi_2 \boldsymbol{b_2}$$

$$\mathcal{A}_{CG} = \int d^2x \left\{ \pi\, g \left(\frac{1}{3} (\nabla \psi_1)^2 + (\nabla \psi_2)^2 \right) + \frac{1}{2} \mathrm{i}\, e_0\, \psi_2\, R \right\}$$

Gaussian free fields $(\nabla \boldsymbol{\Psi})^2$ local curvature

with $\boldsymbol{\Psi} \in \mathbb{R}^2/\mathcal{R}$ where $\mathcal{R} := \mathbb{Z}(\boldsymbol{A} - \boldsymbol{B}) + \mathbb{Z}(\boldsymbol{A} - \boldsymbol{C})$ (repeat lattice)

Effective Coulomb Gas description of FPL on honeycomb



$$g = \frac{1}{\pi}\arccos\left(-\frac{n}{2}\right), \qquad \frac{1}{2} \leq g \leq 1 \qquad (\text{for } 0 \leq n \leq 2)$$

$$4 \leq \kappa = 4/g \leq 8$$

$e_0 = 1 - g$ ($e^{4i\pi\psi_2}$ is marginal)

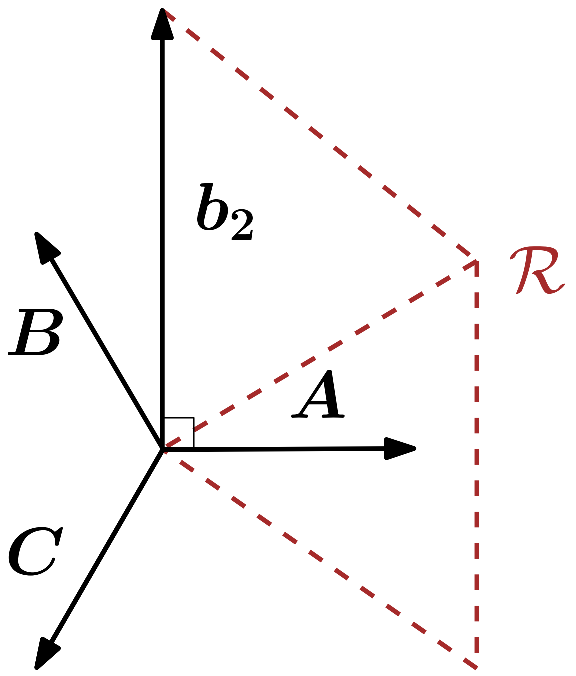
$$\boldsymbol{\Psi} = \psi_1 \boldsymbol{A} + \psi_2 \boldsymbol{b_2}$$

$$\mathcal{A}_{CG} = \int d^2x \left\{ \pi\, g \left(\frac{1}{3} (\nabla\psi_1)^2 + (\nabla\psi_2)^2 \right) + \frac{1}{2} \mathrm{i}\, e_0\, \psi_2\, R \right\}$$

with $\boldsymbol{\Psi} \in \mathbb{R}^2/\mathcal{R}$

$$c_{\mathrm{FPL}}(n) = c_{\mathrm{dense}}(n) + 1 = 2 - 6\frac{(1-g)^2}{g}$$

Effective Coulomb Gas description of ~~FPL~~ dense on honeycomb



$$g = \frac{1}{\pi}\arccos\left(-\frac{n}{2}\right), \qquad \frac{1}{2} \leq g \leq 1 \qquad (\text{ for } 0 \leq n \leq 2\,)$$

$$4 \leq \kappa = 4/g \leq 8$$

$$e_0 = 1 - g$$ ($e^{4\mathrm{i}\pi\psi_2}$ is marginal)

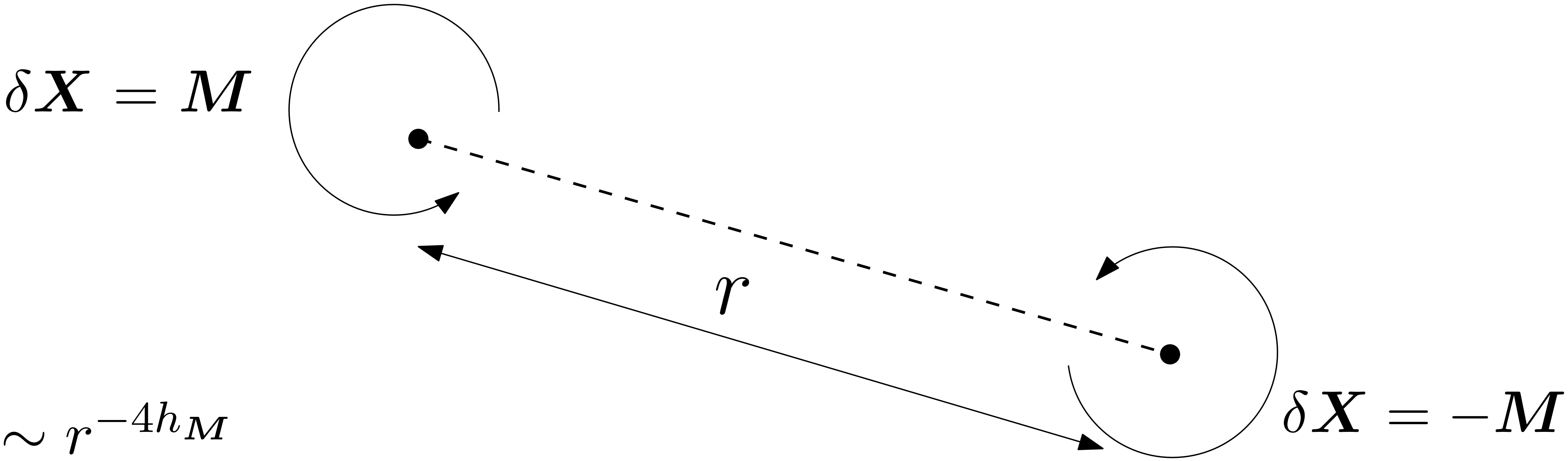
$$\boldsymbol{\Psi} = \cancel{\psi_1 \boldsymbol{A}} + \psi_2 \boldsymbol{b_2}$$

$$\mathcal{A}_{CG} = \int d^2x \left\{ \pi\, g \left(\frac{1}{3}\cancel{(\nabla\psi_1)^2} + (\nabla\psi_2)^2 \right) + \frac{1}{2}\mathrm{i}\, e_0\, \psi_2\, R \right\}$$

with $\boldsymbol{\Psi} \cancel{\in \mathbb{R}^2}/\mathcal{R}$ $\quad \psi_2 \in \mathbb{R}/\mathbb{Z}$

$$c_{\cancel{\mathrm{FPL}}\ \text{dense}}(n) = \cancel{2}\ 1 - 6\frac{(1-g)^2}{g}$$

Correlation of « magnetic operators » = dislocations



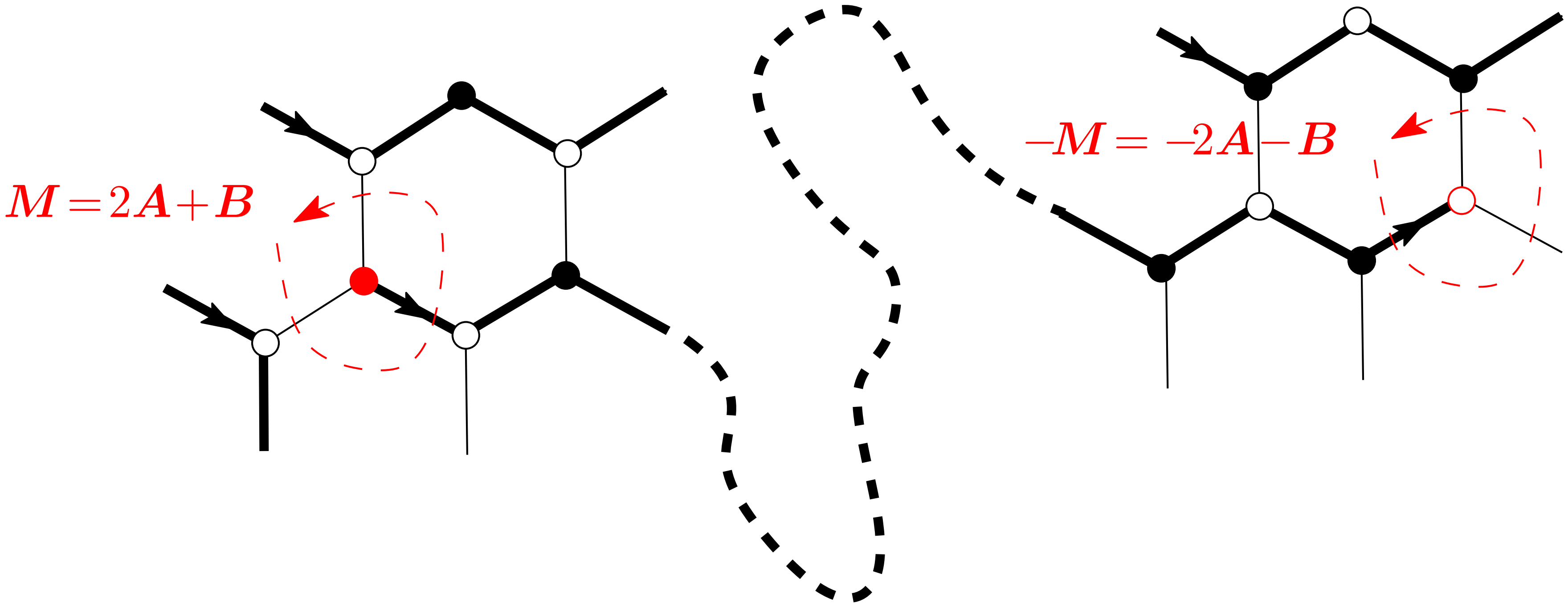
Correlator $\sim r^{-4h_{\boldsymbol{M}}}$

$$h_{\boldsymbol{M}} = \frac{g}{12}\phi_1^2 + \frac{g}{4}\left(1-\delta_{\phi_2,0}\right)\left(\phi_2^2 - \left(1-g^{-1}\right)^2\right) \qquad \text{for} \quad \boldsymbol{M} = \phi_1 \boldsymbol{A} + \phi_2 \boldsymbol{b_2}$$

J. Kondev, J. de Gier, B. Nienhuis 1996

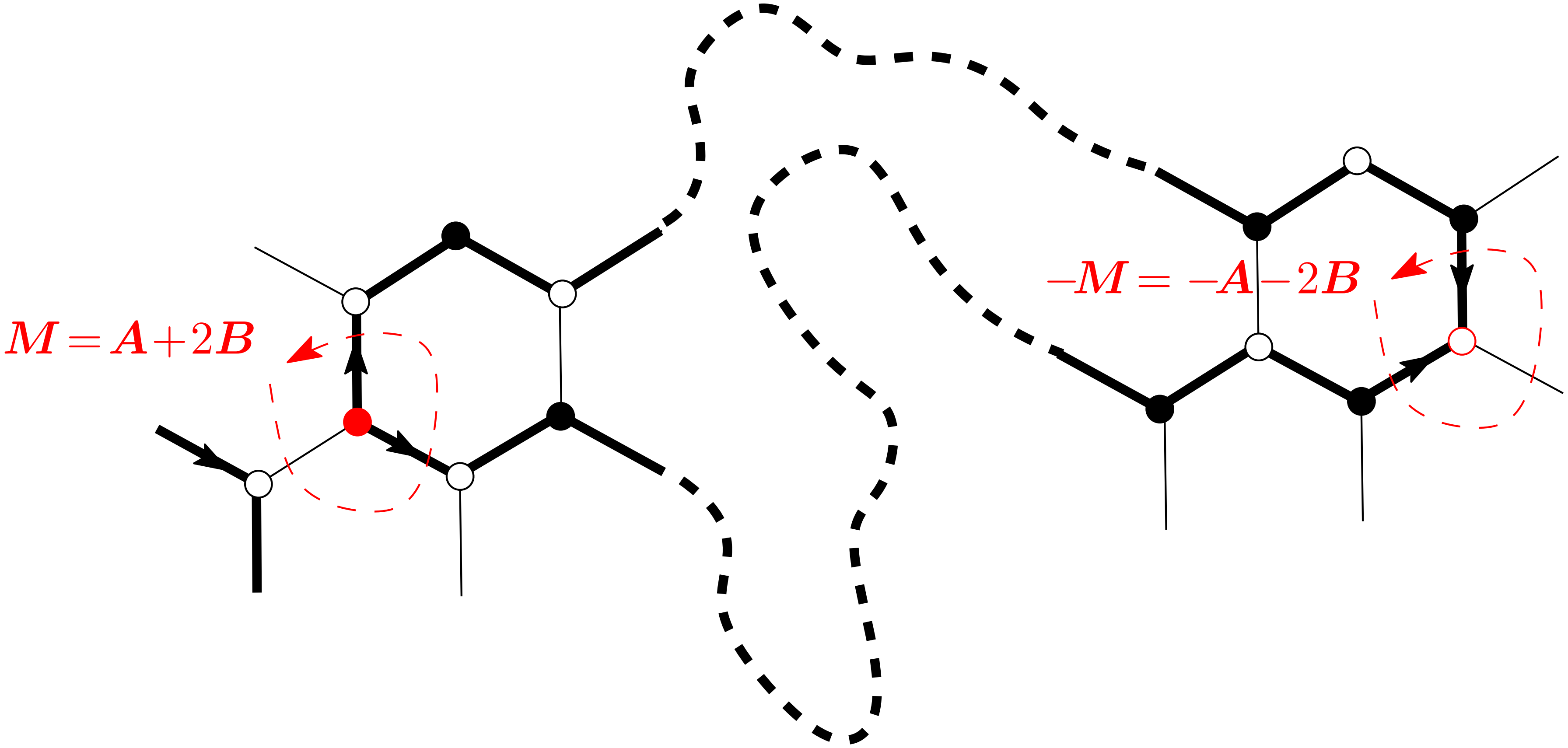
$n = 0$ (i.e., $g = 1/2$): $\quad h_{\boldsymbol{M}}(n=0) = \frac{1}{24}\phi_1^2 + \frac{1}{8}\left(1-\delta_{\phi_2,0}\right)\left(\phi_2^2 - 1\right)$

Examples $\boldsymbol{M} = \boldsymbol{B} + 2\boldsymbol{A} = \frac{3}{2}\boldsymbol{A} + \frac{1}{2}\boldsymbol{b_2}$



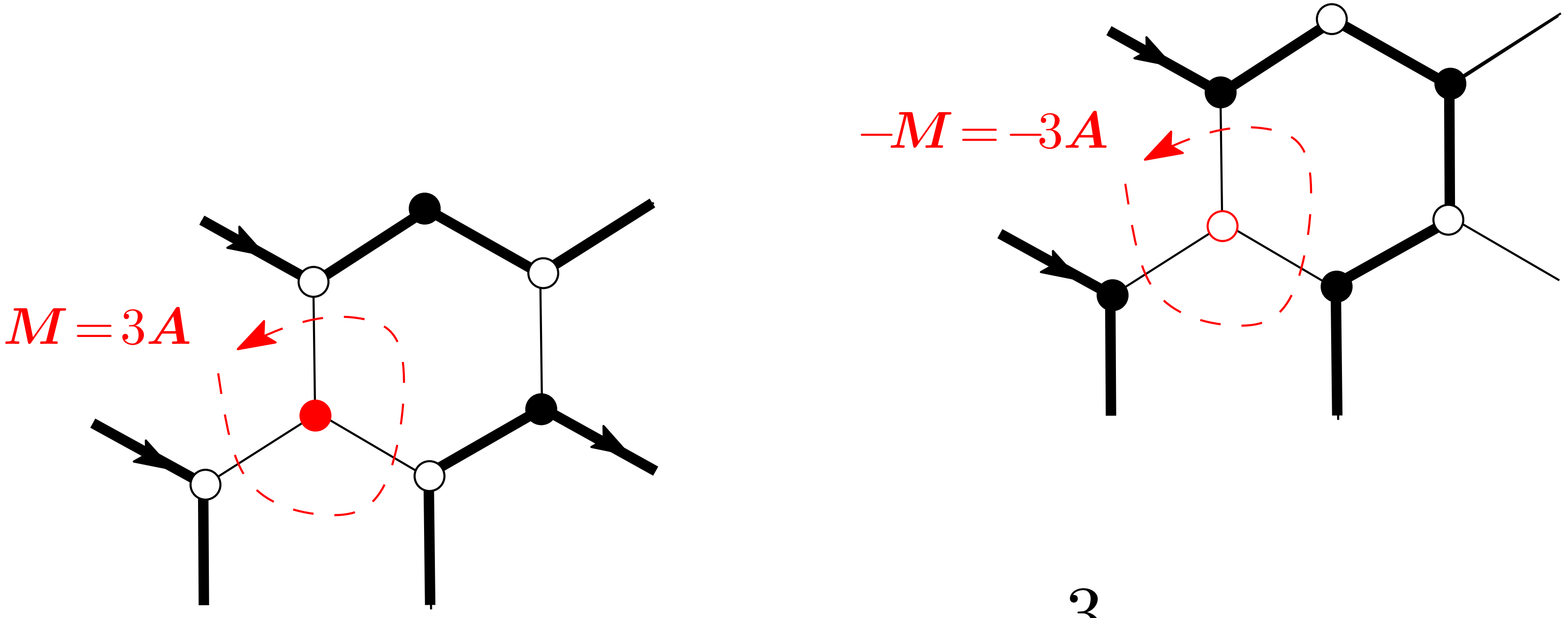
$$h_{\boldsymbol{B}+2\boldsymbol{A}}(n=0) = 0$$

$$M = A + 2B = b_2$$



$$h_{A+2B}(n = 0) = 0$$

$\boldsymbol{M} = 3\boldsymbol{A}$



$$h_{3\boldsymbol{A}}(n=0) = \frac{3}{8}$$

5. KPZ predictions I: partition function

$$\mathcal{Z}_A \sim \text{const.}\ \mu^A A^{\gamma(c)-3} \qquad \gamma(c) = \frac{1}{12}\left(c - 1 - \sqrt{(1-c)(25-c)}\right)$$

$n = 0$ (i.e., $g = 1/2$): $\qquad c_{\text{FPL}}(n=0) = -1$

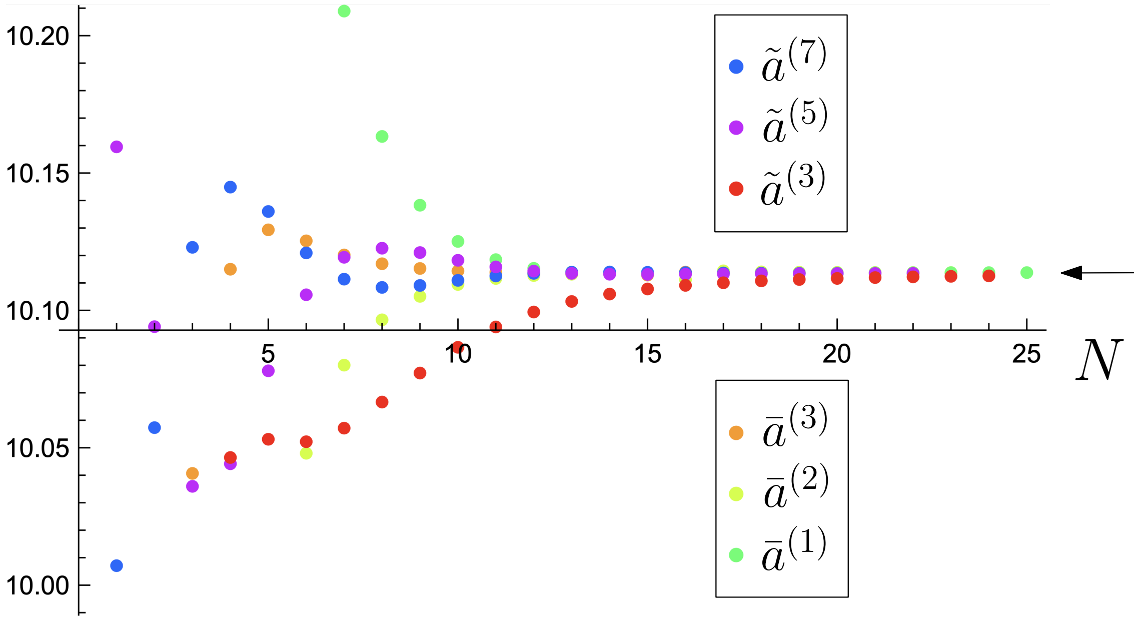
$$A = 2N \qquad z_N = 2N \times \mathcal{Z}_{2N} \sim \text{const.}\ \frac{\mu^{2N}}{N^{2-\gamma}}$$

$$\gamma = \gamma(c=-1) = -\frac{1+\sqrt{13}}{6}$$

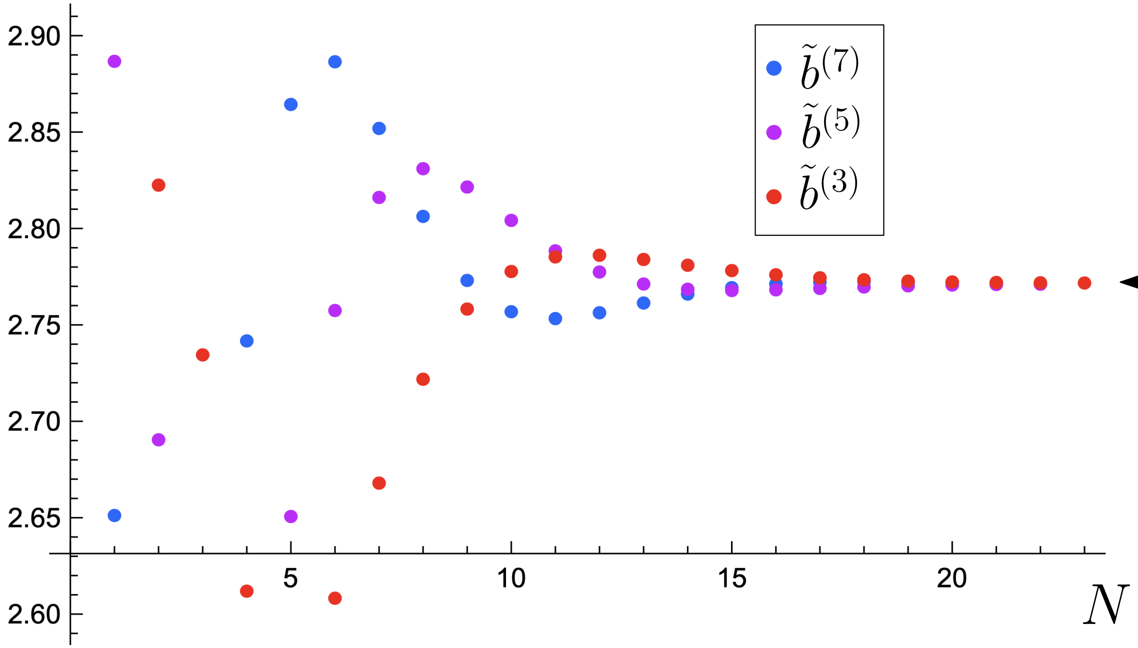
E.Guitter, C. Kristjansen, J. Nielsen 1999

6. Numerics

(Transfer matrix)



$\mu^2 = 10.113$



$2 - \gamma = 2.77$

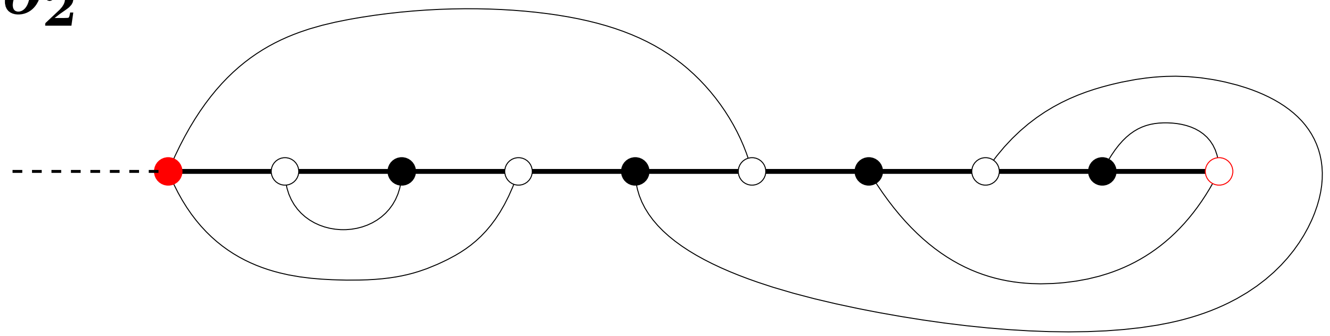
vs prediction

$$2 - \gamma = \frac{13 + \sqrt{13}}{6} = 2.76759\cdots$$

7. KPZ predictions II: correlators

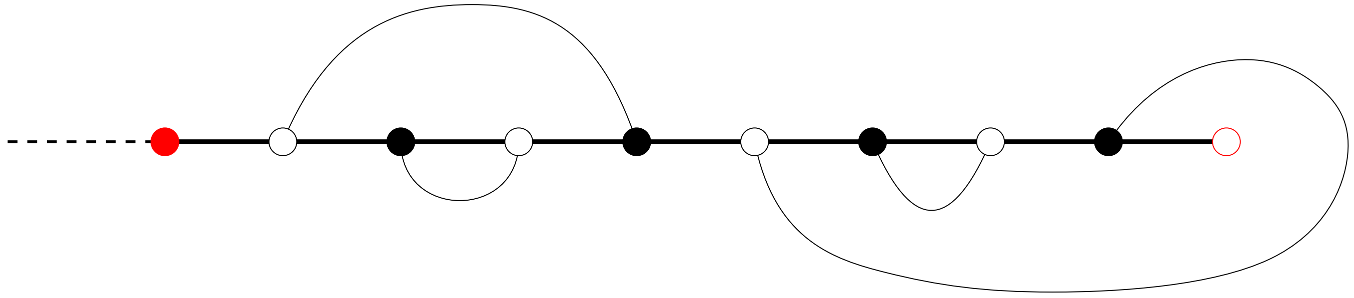
$$\boldsymbol{M} = 2\boldsymbol{A} + \boldsymbol{B} = \frac{3}{2}\boldsymbol{A} + \frac{1}{2}\boldsymbol{b_2}$$

$y_N :=$



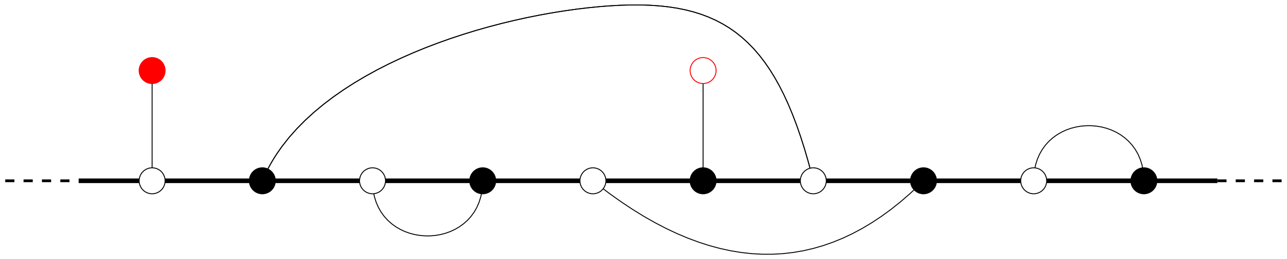
$$\boldsymbol{M} = \boldsymbol{B} = -\frac{1}{2}\boldsymbol{A} + \frac{1}{2}\boldsymbol{b_2}$$

$x_N :=$



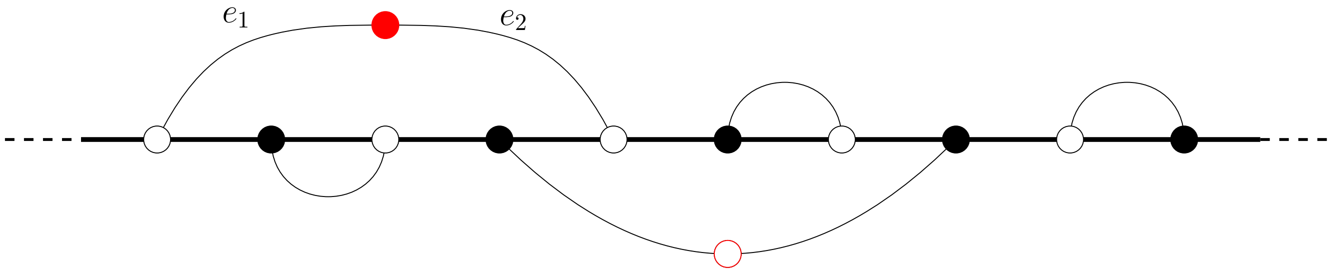
$\boldsymbol{M} = \boldsymbol{A}$

$$w_N := \frac{1}{2} \times$$



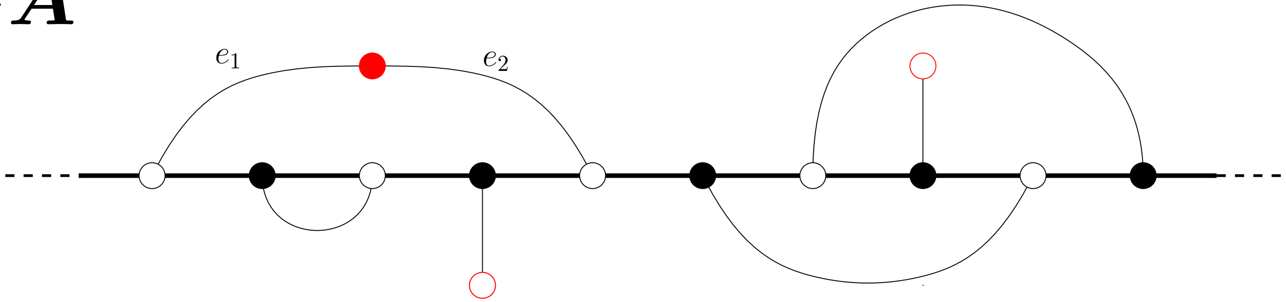
$\boldsymbol{M} = 2\boldsymbol{A}$

$$v_N :=$$



$2\boldsymbol{A}$, $-\boldsymbol{A}$ and $-\boldsymbol{A}$

$$u_N := \frac{1}{4} \times$$



$$\beta_z = 2 - \gamma \ , \qquad \beta_y = 1 + 2\Delta_{\frac{3}{2}\boldsymbol{A}+\frac{1}{2}\boldsymbol{b_2}} - \gamma \ , \qquad \beta_x = 1 + 2\Delta_{-\frac{1}{2}\boldsymbol{A}+\frac{1}{2}\boldsymbol{b_2}} - \gamma$$

$$\beta_w = 1 + 2\Delta_{\boldsymbol{A}} - \gamma \ , \qquad \beta_v = 1 + 2\Delta_{2\boldsymbol{A}} - \gamma \ , \qquad \beta_u = \Delta_{2\boldsymbol{A}} + 2\Delta_{\boldsymbol{A}} - \gamma \ .$$

$$\Delta_{\boldsymbol{M}} := \Delta(h_{\boldsymbol{M}}, -1) = \frac{\sqrt{1 + 12h_{\boldsymbol{M}}} - 1}{\sqrt{13} - 1}$$

$$h_{\boldsymbol{M}} = \frac{1}{24}\phi_1^2 + \frac{1}{8}\left(1 - \delta_{\phi_2,0}\right)\left(\phi_2^2 - 1\right) \qquad \text{for} \quad \boldsymbol{M} = \phi_1 \boldsymbol{A} + \phi_2 \boldsymbol{b_2}$$

$$\beta_z = 2-\gamma\ , \qquad \beta_y = 1+2\Delta_{\frac{3}{2}\boldsymbol{A}+\frac{1}{2}\boldsymbol{b_2}} - \gamma\ , \qquad \beta_x = 1+2\Delta_{-\frac{1}{2}\boldsymbol{A}+\frac{1}{2}\boldsymbol{b_2}} - \gamma$$

$$\beta_w = 1+2\Delta_{\boldsymbol{A}} - \gamma\ , \qquad \beta_v = 1+2\Delta_{2\boldsymbol{A}} - \gamma\ , \qquad \beta_u = \Delta_{2\boldsymbol{A}} + 2\Delta_{\boldsymbol{A}} - \gamma\ .$$

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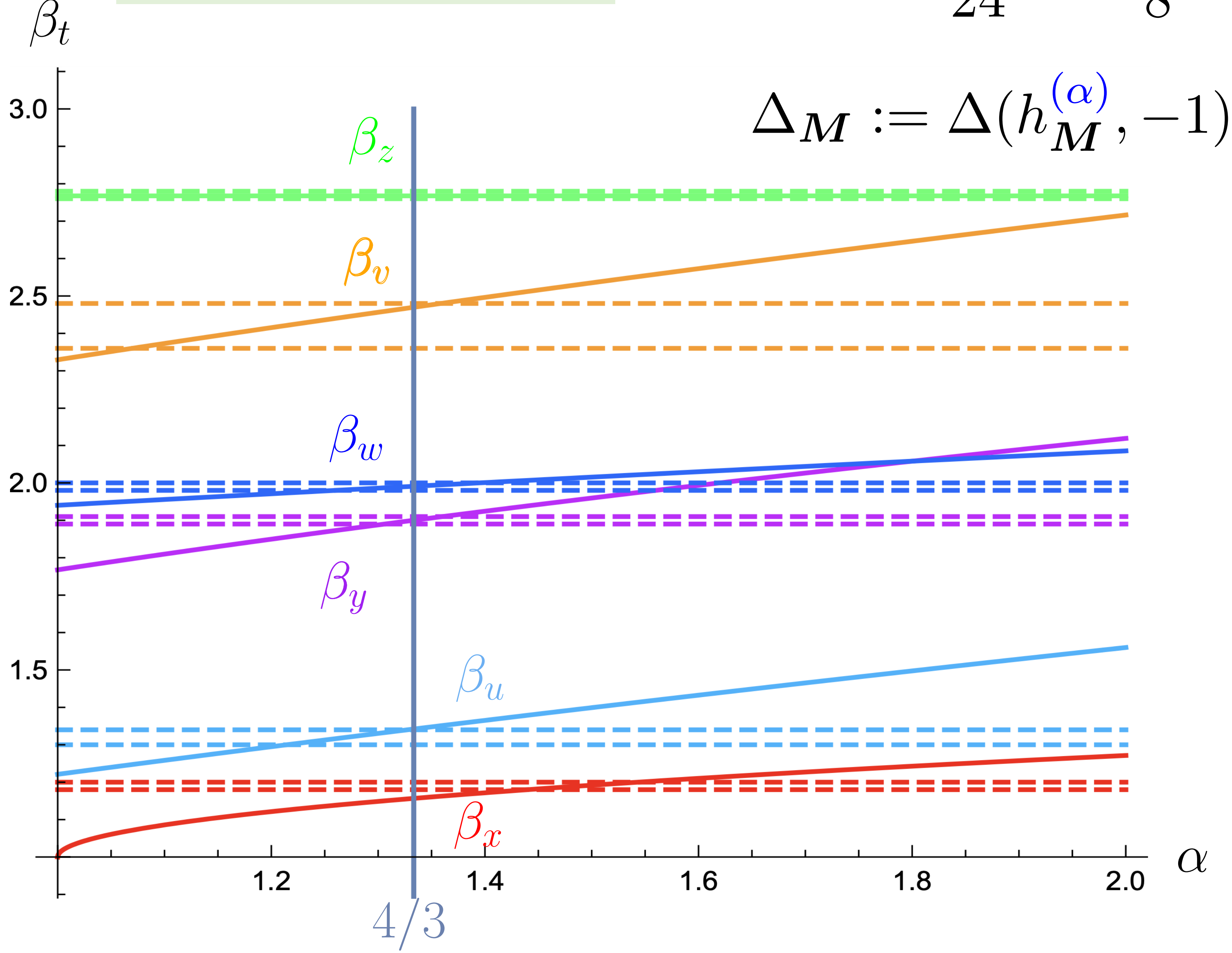
$$h_{\boldsymbol{M}} = \frac{1}{24}\phi_1^2 + \frac{1}{8}\left(1-\delta_{\phi_2,0}\right)\left(\phi_2^2-1\right) \qquad \text{for} \quad \boldsymbol{M} = \phi_1\boldsymbol{A} + \phi_2\boldsymbol{b_2}$$

	numerics	KPZ	
β_z	2.77 ± 0.01	$\frac{1}{6}\left(13+\sqrt{13}\right) = 2.76759\ldots$	✓
β_y	1.90 ± 0.01	$\frac{1}{6}\left(7+\sqrt{13}\right) = 1.76759\ldots$	Discrepancy
β_x	1.19 ± 0.01	1	Discrepancy
β_w	1.99 ± 0.01	$1+\frac{\sqrt{6}}{\sqrt{13}-1} = 1.94010\ldots$	Discrepancy
β_v	2.42 ± 0.06	$1+\frac{2\sqrt{3}}{\sqrt{13}-1} = 2.32951\ldots$	Discrepancy
β_u	1.32 ± 0.02	$\frac{\sqrt{3}+\sqrt{6}-1}{\sqrt{13}-1} = 1.22106\ldots$	Discrepancy

Renormalization

$$h_{\boldsymbol{M}}^{(\alpha)} = \frac{\alpha}{24}\phi_1^2 + \frac{1}{8}\left(1-\delta_{\phi_2,0}\right)\left(\phi_2^2-1\right)$$

$$\Delta_{\boldsymbol{M}} := \Delta(h_{\boldsymbol{M}}^{(\alpha)}, -1) = \frac{\sqrt{1+12h_{\boldsymbol{M}}^{(\alpha)}}-1}{\sqrt{13}-1}$$



$\alpha \simeq 4/3$

	numerics	(4/3)-corrected KPZ	
β_z	2.77 ± 0.01	$\frac{1}{6}\left(13+\sqrt{13}\right) = 2.76759\ldots$	✓
β_y	1.90 ± 0.01	$1+\frac{\sqrt{22}}{2\left(\sqrt{13}-1\right)} = 1.90008\ldots$	✓
β_x	1.19 ± 0.01	$1+\frac{\sqrt{6}}{6\left(\sqrt{13}-1\right)} = 1.15668\ldots$	$\simeq$
β_w	1.99 ± 0.01	$1+\frac{2\sqrt{15}}{3(\sqrt{13}-1)} = 1.99096\ldots$	✓
β_v	2.42 ± 0.06	$1+\frac{2\sqrt{33}}{3(\sqrt{13}-1)} = 2.46983\ldots$	✓
β_u	1.32 ± 0.02	$\frac{2\sqrt{15}+\sqrt{33}-3}{3\left(\sqrt{13}-1\right)} = 1.34207\ldots$	✓

We used two GFFs $\frac{g'}{3}\,(\nabla\psi_1)^2 + g\,(\nabla\psi_2)^2$

where $g' = g = \frac{1}{\pi}\arccos\left(-\frac{n}{2}\right)\ ,\ \frac{1}{2} \le g \le 1$

- $g :\ e^{4\mathrm{i}\pi\psi_2}$ marginal
- what fixes g' ?

$g = 1 \to g' = g = 1$ (from 3-color symmetry)

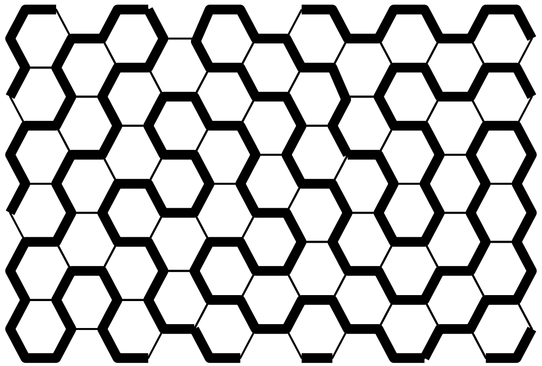
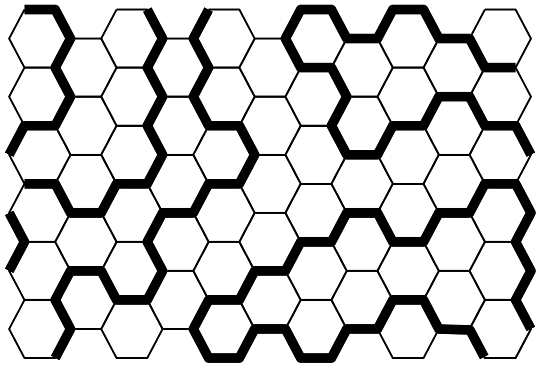
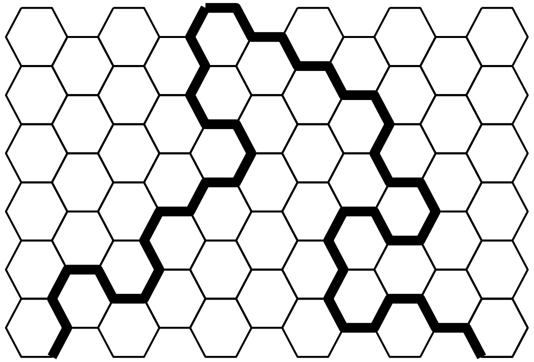
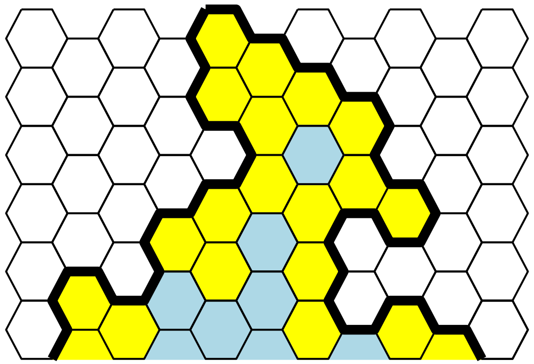
$g' = \alpha(g)\,g$

$g = 1/2 \to \alpha(1/2) = 4/3 \to g' = \alpha g = 2/3$

$g = 2/3 \to \alpha(2/3) = 9/8 \to g' = \alpha g = 3/4$

It is tempting to conjecture $g' = \frac{1}{2-g}$ $\frac{4}{\kappa} + \frac{\kappa'}{4} = 2$

(6-vertex model)
I. Kostov 2000

<table>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td colspan="2">g</td><td>$\tilde{g} = 2 - g$</td><td>$g' = 1/\tilde{g} = \frac{1}{2-g}$</td></tr>
<tr><td colspan="3">$n = -2\cos(\pi g) = -2\cos(\pi \tilde{g})$</td><td>$n' = -2\cos(\pi g')$</td></tr>
<tr><td>$1 + c(g)$</td><td>$c(g) := 1 - 6\frac{(1-g)^2}{g}$</td><td colspan="2">$c(\tilde{g}) = c(g')$</td></tr>
<tr><td>fully packed</td><td>dense</td><td>dilute ← duality</td><td>→ dense</td></tr>
</table>

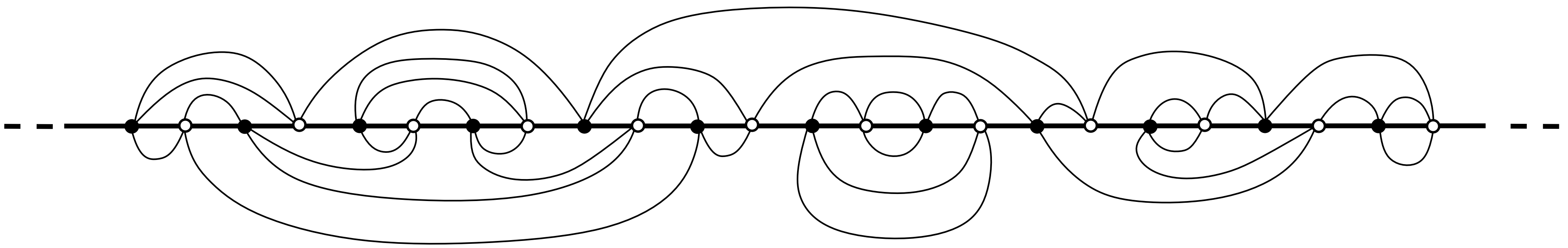
It is tempting to conjecture

$$g' = \frac{1}{2-g}$$

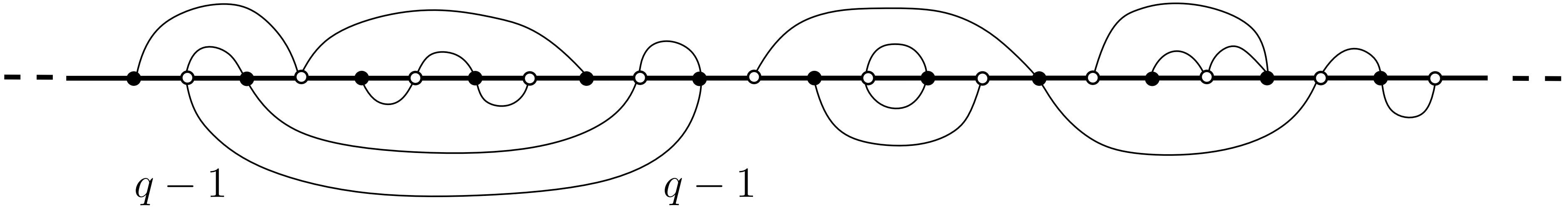
$$\frac{4}{\kappa} + \frac{\kappa'}{4} = 2$$

8. General bicolored maps

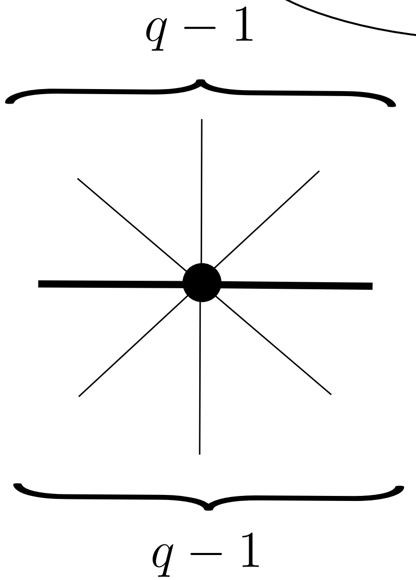
5-regular

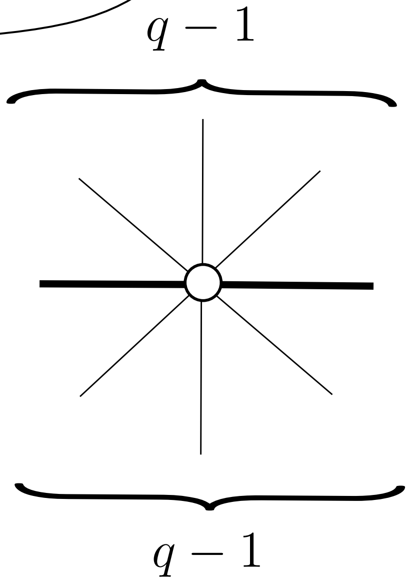


{3,4}-mixed



$q = 4$ rigid





J. Borga, E. Gwynne, X. Sun 2022

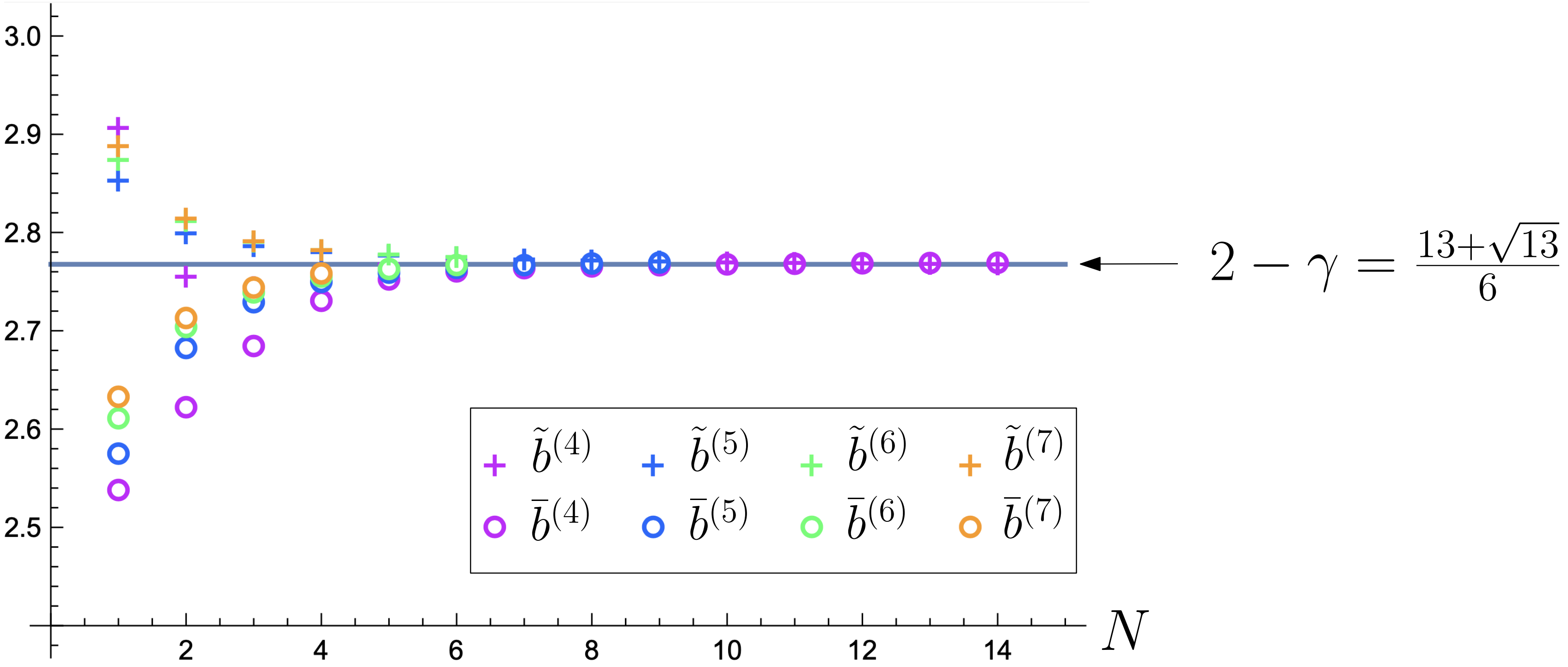
J. Borga, E. Gwynne, M. Park 2022

$q = 2$

4-,5-,6-,7-regular

$$\gamma = \gamma(-1) = -\frac{1+\sqrt{13}}{6}$$

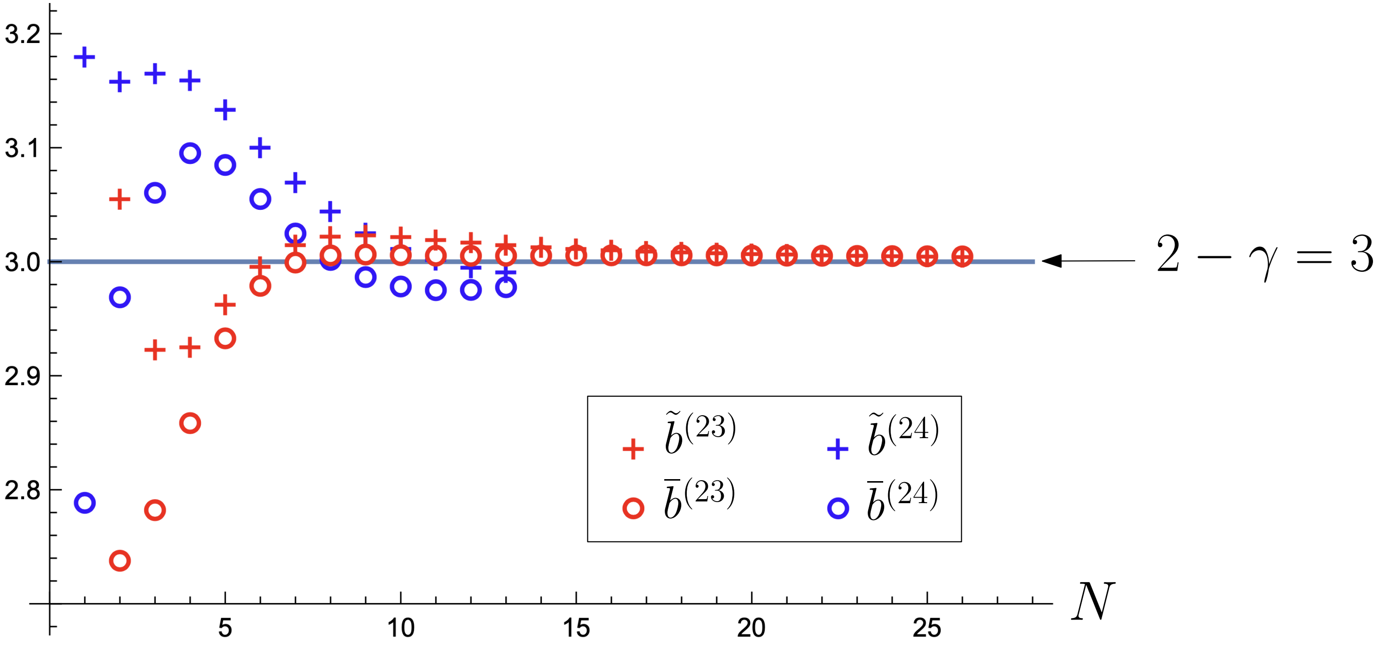
$$c = -1$$



{2,3}-, {2,4}-mixed
& rigid (exact)

$$\gamma = \gamma(-2) = -1$$

$$c = -2$$



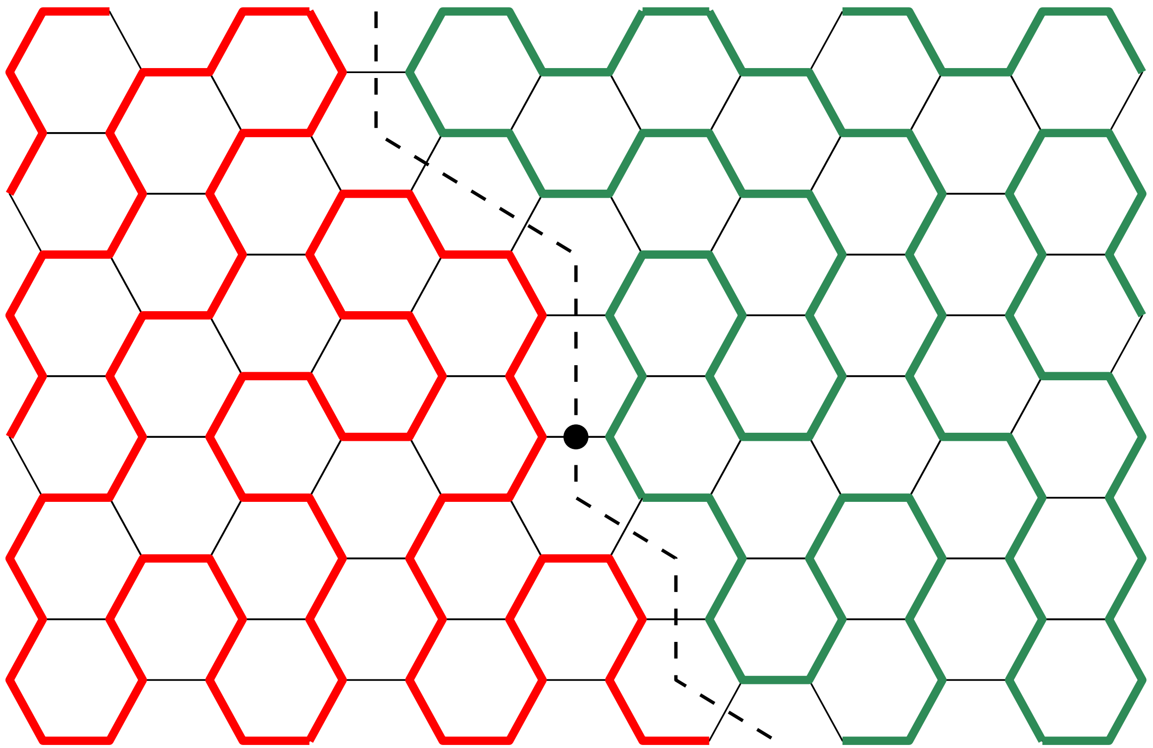
9. Long-distance contacts

$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ SLE$_8$

Hausdorff dimension $D = 2$

$\widetilde{\mathcal{C}} = \mathcal{C}_1 \cap \mathcal{C}_2$ SLE$_2$

Hausdorff dimension $\widetilde{D} = 5/4$

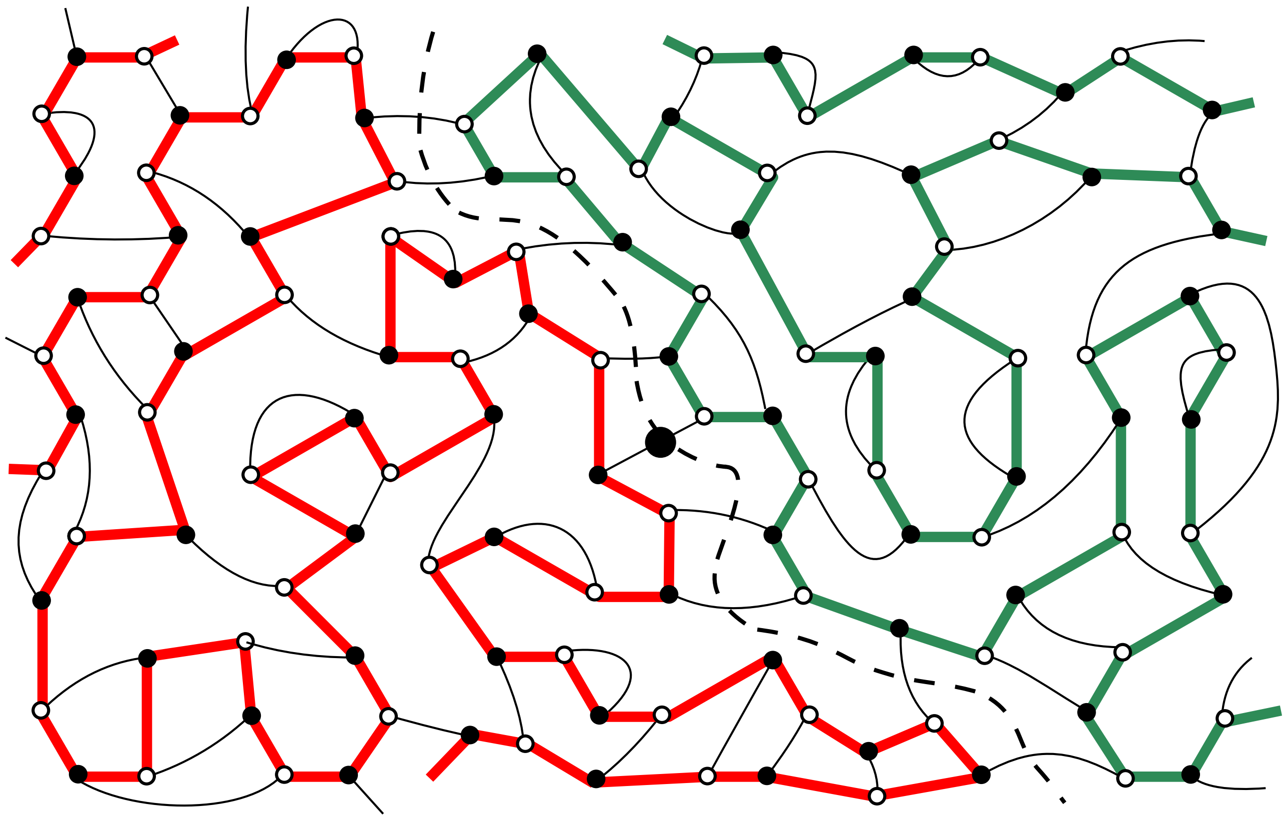


$$\mathbb{E}\,|\mathcal{C}_1 \cap \mathcal{C}_2| \asymp A^{\widetilde{D}/2} = A^{1-h_{1\cap 2}}, \quad h_{1\cap 2} = 3/8, \quad A \to \infty$$

Liouville Quantum Gravity

$$\gamma_{\mathrm{L}} = \sqrt{2}, \quad c = -2$$

$$\gamma_{\mathrm{L}} = \frac{1}{\sqrt{3}}\left(\sqrt{13} - 1\right), \ c = -1$$



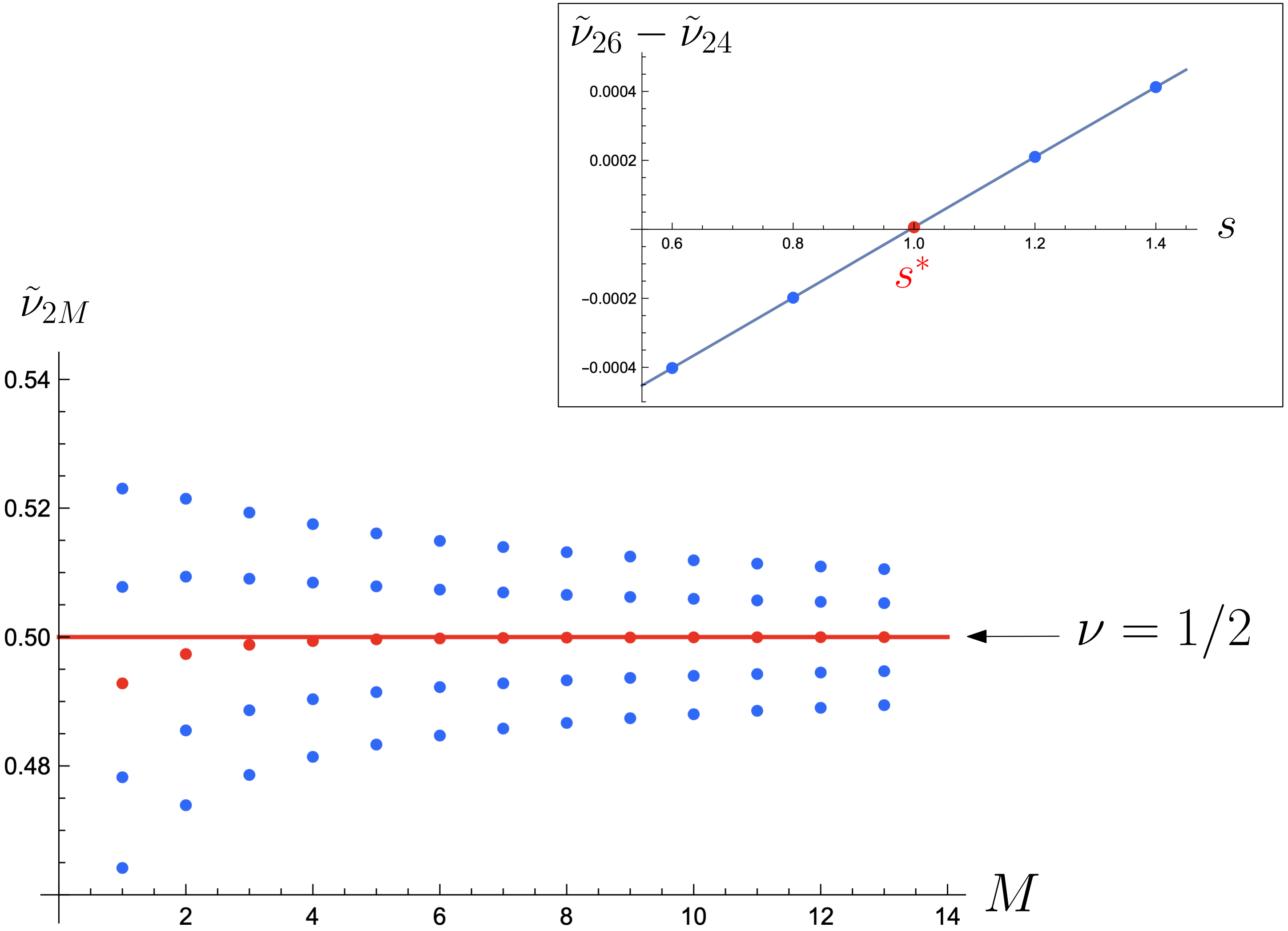
$$\mathbb{E}_{\mathrm{LQG}}|\mathcal{C}_1 \cap \mathcal{C}_2| \asymp \mathcal{A}^{\nu} := \mathcal{A}^{1-\Delta_{1\cap 2}}$$

$$\Delta_{1\cap 2} = \Delta(3/8, c = -2) = 1/2 \ ,$$

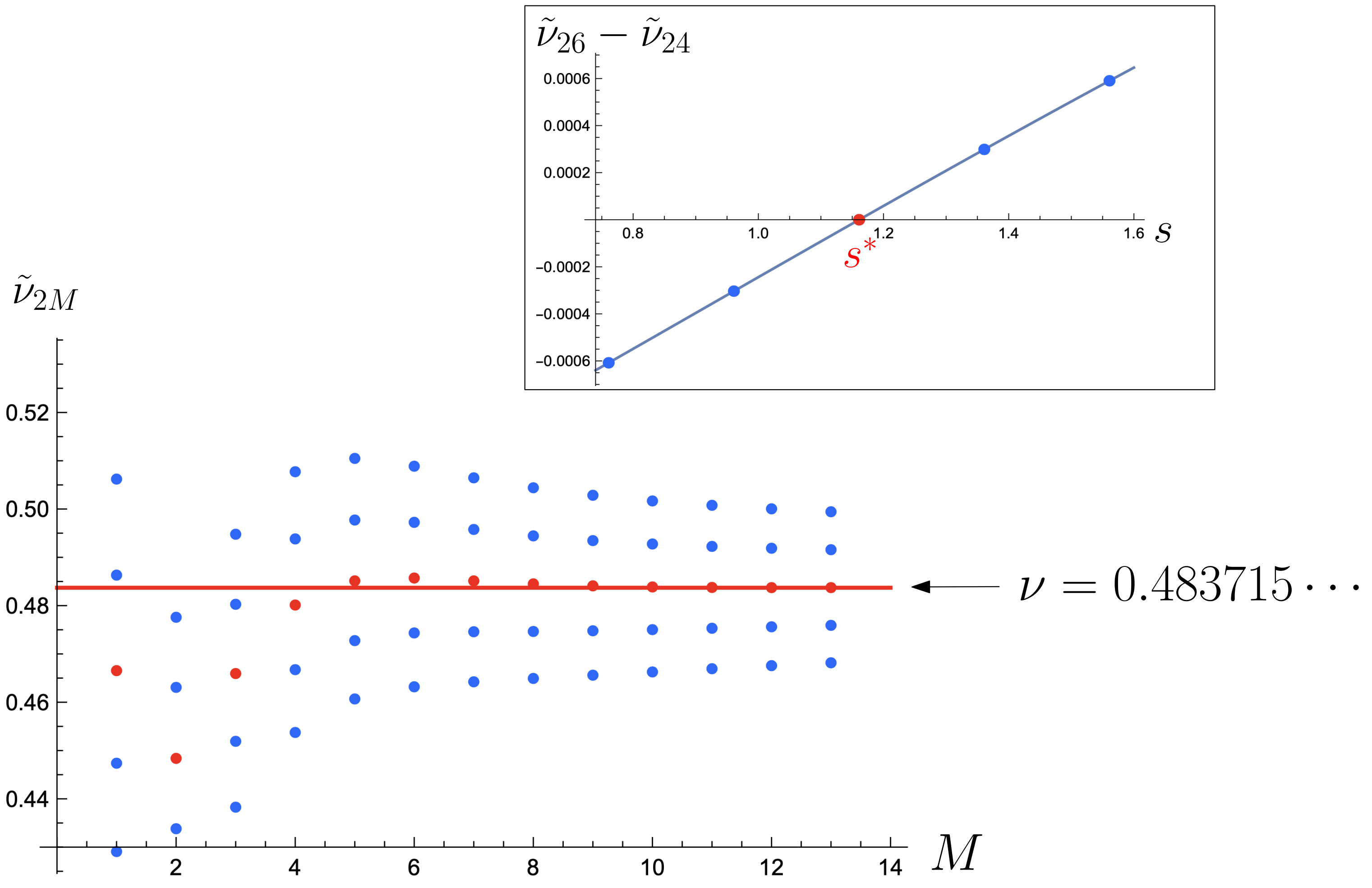
$$\nu = 1 - \Delta_{1\cap 2} = 1/2 \ ;$$

$$\Delta_{1\cap 2} = \Delta(3/8, c = -1) = \frac{\sqrt{11} - \sqrt{2}}{\sqrt{26} - \sqrt{2}} \ ,$$

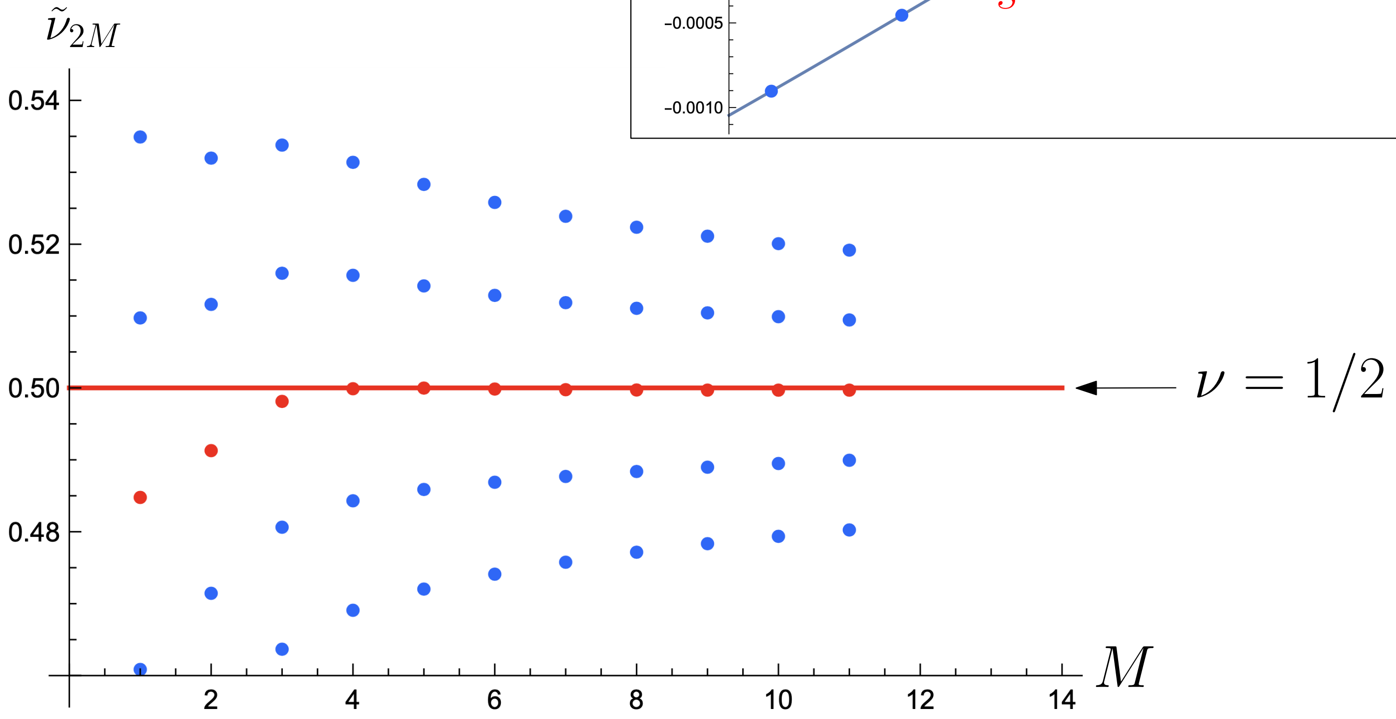
$$\nu = 1 - \Delta_{1\cap 2} = \frac{\sqrt{26} - \sqrt{11}}{\sqrt{26} - \sqrt{2}} = 0.483715$$

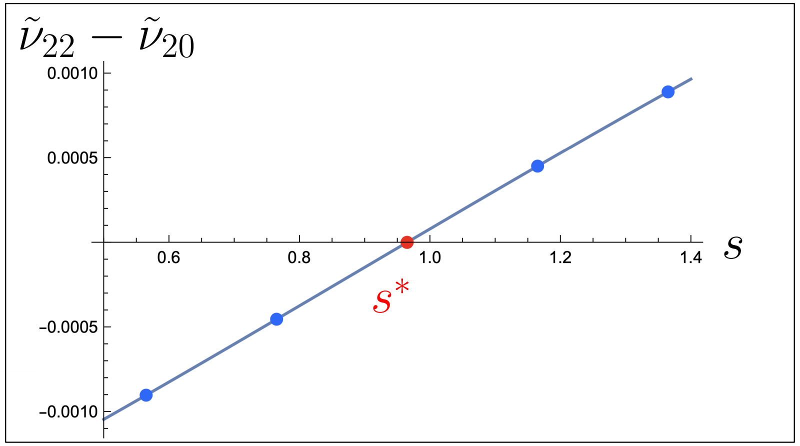


exponent ν for rigid Hamiltonian cycles on 4-regular bicolored maps



exponent ν for Hamiltonian cycles on 3-regular bicolored maps





Hamiltonian cycles on bicolored maps with mixed valencies 2 and 3

SLE vs fully packed exponents

$$\kappa = \frac{4\pi}{\arccos(-n/2)} \in (4, 8] \quad \text{for} \quad n \in [0, 2)$$

$$h_\ell^{(\kappa)} = \frac{1}{16\kappa}\left[4\ell^2 - (4-\kappa)^2\right], \quad \ell \in \mathbb{Z}^+$$

(multiple SLEs, arm exponents)

$$h_{2k}^{\mathrm{fpl}(n)} = h_{2k}^{(\kappa)},$$

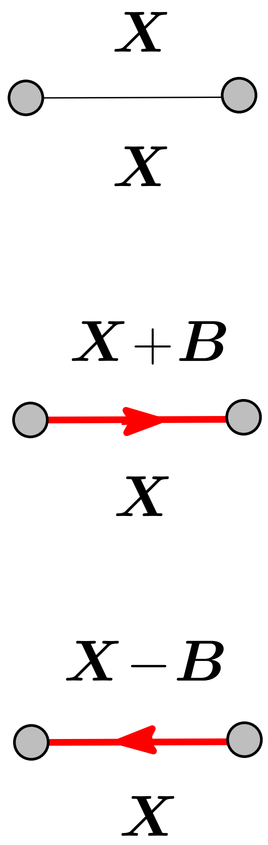
$$h_{2k-1}^{\mathrm{fpl}(n)} = h_{2k-1}^{(\kappa)} + \frac{3}{4\kappa} \quad (\hexagon),$$

$$h_{2k-1}^{\mathrm{fpl}(n)} = h_{2k-1}^{(\kappa)} + \frac{1}{6+\kappa} \quad (\square), \quad k \in \mathbb{Z}^+.$$

$$h_{1\cap 2} := h_{\ell=4}^{\mathrm{fpl}(0)} = h_{\ell=4}^{(\kappa=8)} = h_{\ell=2}^{(\widetilde{\kappa}=2)} = \frac{3}{8}$$

Thank you!

On the importance of being bicolored   FPL(0) model on cubic planar maps?



$$\boldsymbol{A}=\mathbf{0} \qquad \boldsymbol{C}=-\boldsymbol{B}$$

$$c_{\text{dense}}(n=0)=-2$$ B.D., I. Kostov 1988

$$z_N^\circ \sim \text{const.}\ \frac{(\mu^\circ)^{2N}}{N^{2-\gamma^\circ}} \qquad \gamma^\circ=\gamma(c=-2)=-1$$

$$z_N^\circ=\sum_{k=0}^{N}\binom{2N}{2k}\mathrm{Cat}_k\mathrm{Cat}_{N-k}=\mathrm{Cat}_N\mathrm{Cat}_{N+1}\sim \text{const.}\ \frac{4^{2N}}{N^3}$$

where $\mathrm{Cat}_N=\binom{2N}{N}/(N+1)$