UNIVERSITY OF BIRMINGHAM | SCHOOL OF MATHEMATICS

MARIE CURIE ACTIONS

LEVERHULME TRUST

Geometry and Borel Summability of Exact WKB Solutions

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Invitation to Recursion, Resurgence and Combinatorics

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Okinawa, Japan

§0. Setting

- Start with singularly perturbed linear ODE in a domain $\mathsf{X} \subset \mathbb{C}_x$:

$$(\hbar\partial_x)^n\psi + p_1(\hbar\partial_x)^{n-1}\psi + \ldots = \left(\sum_{k=0}^{n} p_{n-k}\hbar^k\partial_x^k\right)\psi(x,\hbar) = 0 \qquad (\bigstar)$$

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Two Questions Addressed Today

1. When does the WKB method lead to solutions of $(\bigstar)$ with *good* asymptotics as $\hbar \to 0$?
2. What is the WKB method for P and ∇?

§1.1. Formal WKB Method (Quick Reminder)

- Plug the WKB ansatz into ($\bigstar$) to get a nonlinear ODE of order $n-1$:

$$(\hbar\partial_x)^{n-1}s + s^n + \ldots = 0\ ; \qquad \text{explicitly:} \quad \sum_{k=1}^{n} p_k(\hbar\partial_x + s)^{k-1}s = 0 \qquad (\blacklozenge)$$

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Formal Existence and Uniqueness Theorem [classical]

If the basepoint x_0 is chosen *generically*, there are n formal solutions

$$\widehat{s}_i(x,\hbar) = \sum_{k=0}^{\infty} s_i^{(k)}(x)\hbar^k \in \mathcal{O}_{\mathsf{X},x_0}[\![\hbar]\!] \qquad i = 1,\ldots,n$$

uniquely and recursively determined by leading-orders $s_i^{(0)} = \lambda_i(x)$ that are roots of

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- "Generically" := away from ***turning points*** := zeros of the discriminant of (♠)
- $\widehat{\psi}_k$ is very computable but almost always **divergent!**

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3. Geometrically, the WKB method is a method to search for an invariant splitting of an oper structure on $(\mathcal{E}, \nabla)$, so exact WKB solutions make sense for connections.

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- ***WKB trajectory of type ij*** emanating from x_0 is locally given by

$$\Gamma_{ij}(x_0) \quad : \quad \mathrm{Im}\left(\int_{x_0}^{x} (\lambda_i - \lambda_j)\,\mathrm{d}x\right) = 0 \quad \text{and} \quad \mathrm{Re}\left(\int_{x_0}^{x} (\lambda_i - \lambda_j)\,\mathrm{d}x\right) \geqslant 0$$

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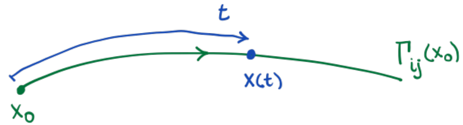
- Natural flow time parameter: $t(x) := \displaystyle\int_{x_0}^{x(t)} (\lambda_i - \lambda_j)\,\mathrm{d}x$



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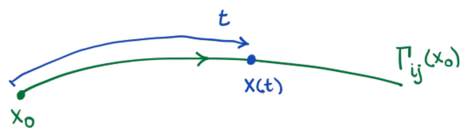
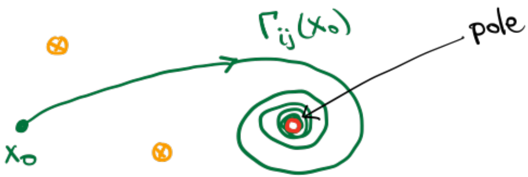
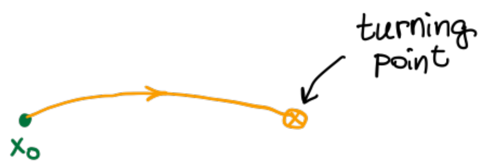


- $\Gamma_{ij}(x_0)$ is ***nonsingular*** if it is infinitely long and encounters no turning points
- $\Gamma_{ij}(x_0)$ is ***singular*** if it flows into a turning point





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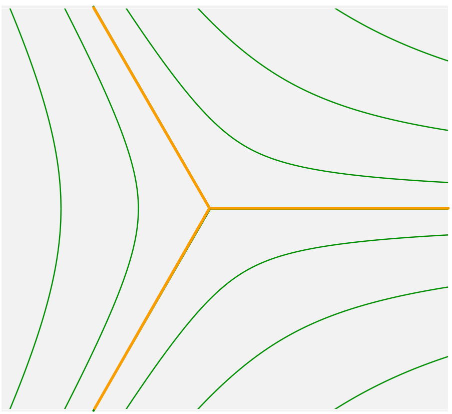
- ***Stokes 'graph'*** or ***network*** := collection of all Stokes lines on X



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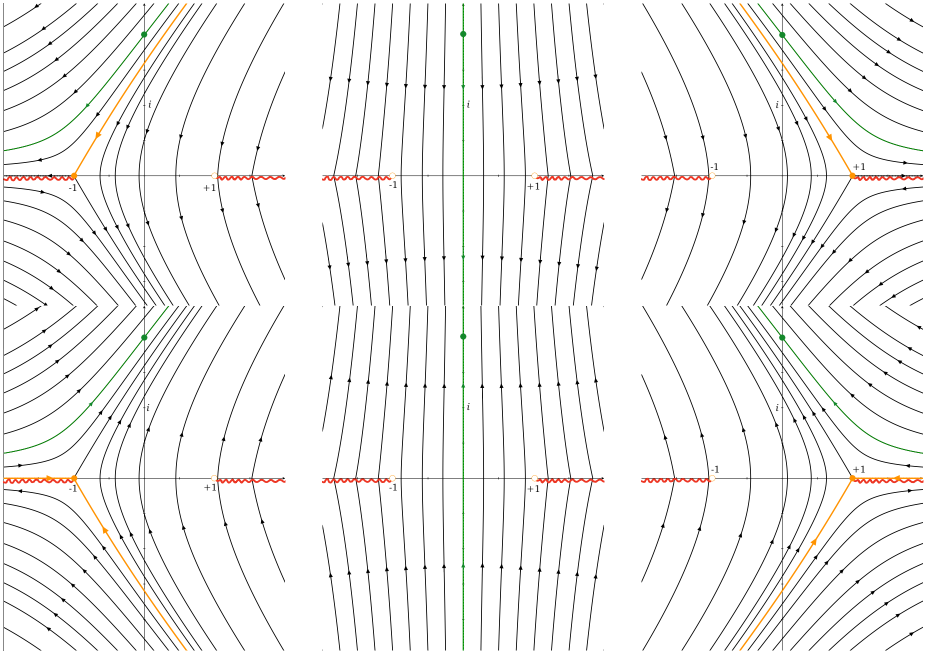

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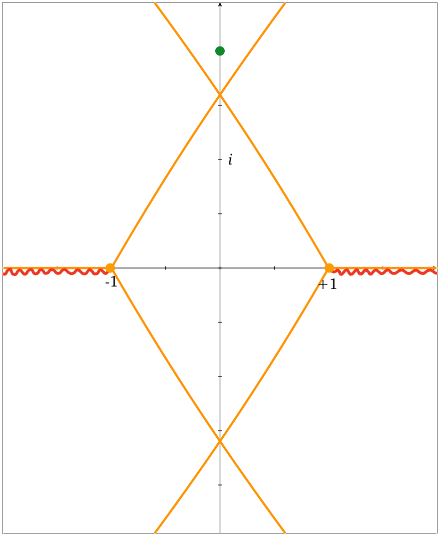

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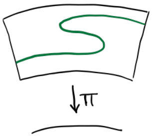

$$\begin{array}{ccc} \Sigma & \hookrightarrow & \mathsf{T}^*_{\mathsf{X},\mathsf{D}} = \mathsf{tot}\left(\Omega^1_{\mathsf{X},\mathsf{D}}\right) \\ \downarrow \pi & \swarrow & \\ \mathsf{X} & & \end{array}$$

equipped with canonical differential
$\lambda \in \mathsf{H}^0_\Sigma(\Omega^1_{\Sigma,\tilde{\mathsf{D}}})$

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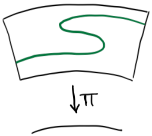

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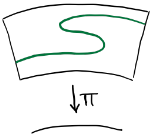

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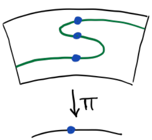

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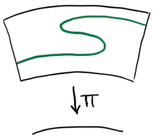

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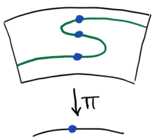

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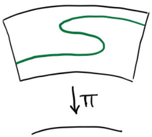

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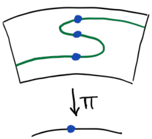

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- Stokes network on X is the projection of the Stokes graph under $\mathrm{ad}\,\pi : \mathrm{ad}\,\Sigma \longrightarrow \mathsf{X}$

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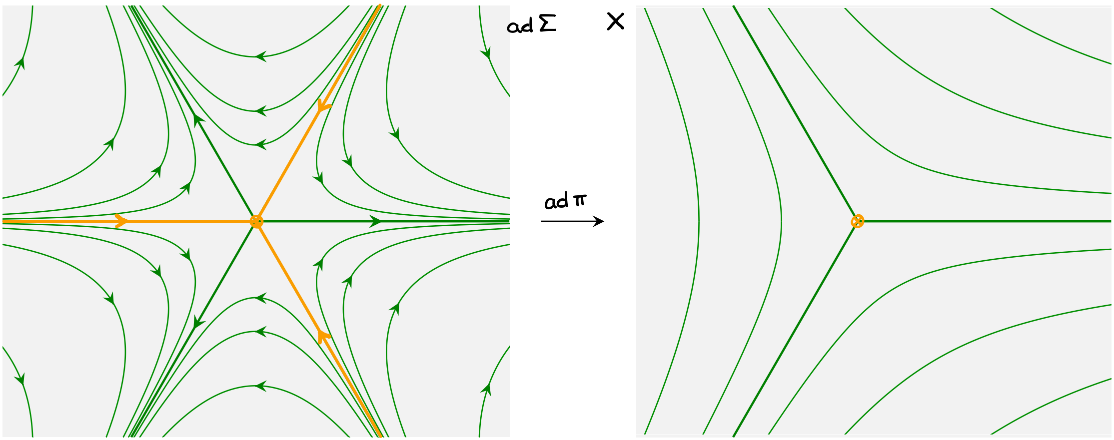
Definition ($n = 2$)

The ***WKB flow of x_0 of type i is nonsingular*** if the WKB trajectory $\Gamma_{ij}(x_0)$ is nonsingular.



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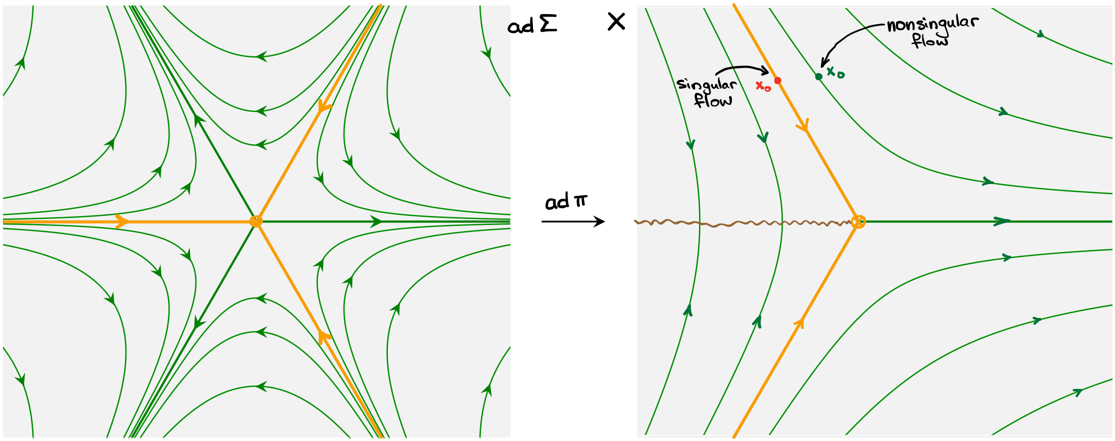

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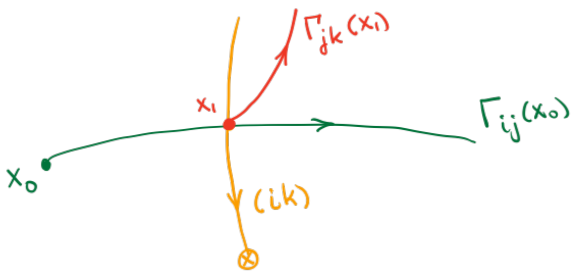

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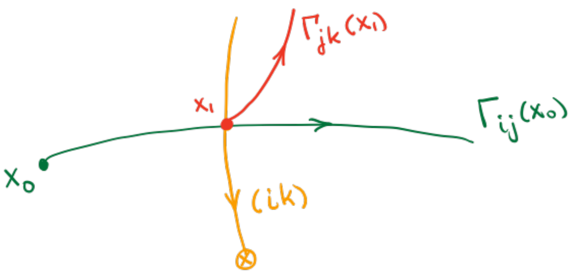

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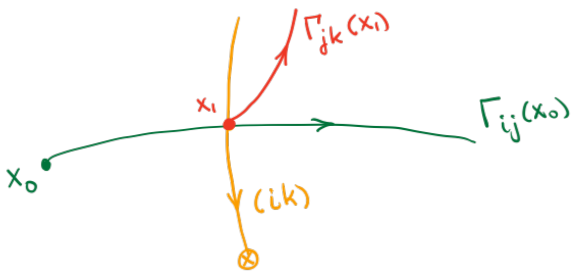

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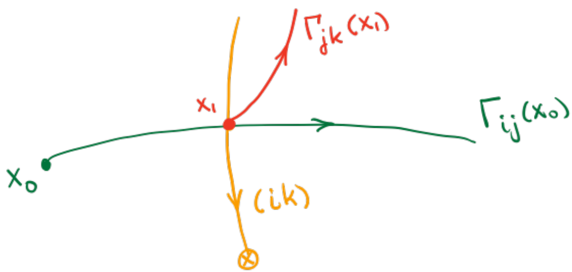
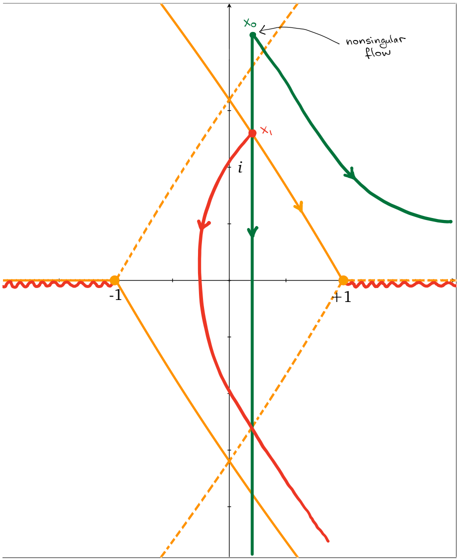
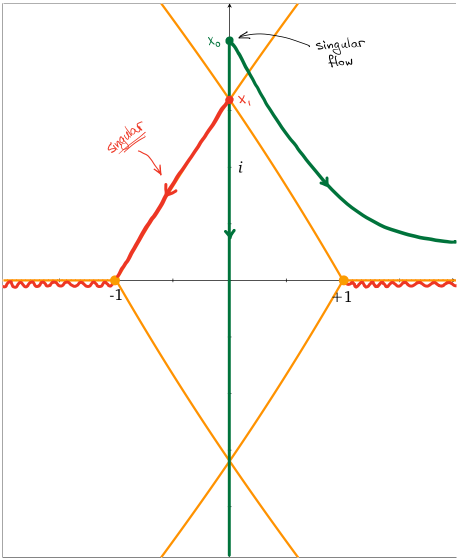


- ***Complete Stokes network*** := locus of all points on X with singular WKB flow

Example (BNR): $(\hbar^3\partial_x^3 + 3\hbar\partial_x + 2ix)\psi = 0$





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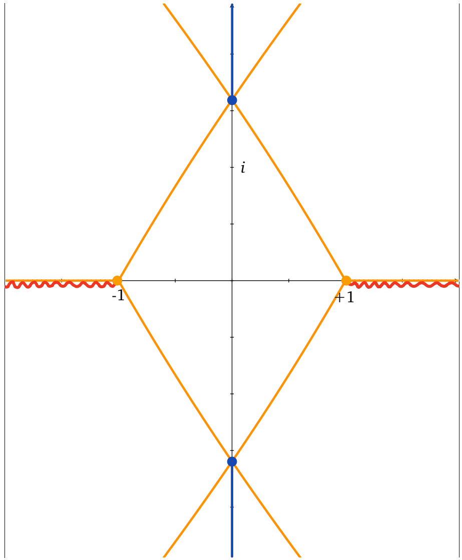

§3. Borel Summability of WKB Solutions

Theorem [N] (Existence and Uniqueness of Exact WKB Solutions)

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Then the formal WKB solution

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In fact, ψ_i is the unique solution for x near x_0 which is asymptotically smooth with factorial growth uniformly as $\hbar \to 0$ with $\mathrm{Re}(\hbar) > 0$ and uniformly in x, and satisfies

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Corollary

Uniqueness yields a notion of ***exact WKB flat sections*** of $\mathcal{L}$ for P on (X, D).

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Goal

Construct the analytic continuation $\boldsymbol{\sigma}_{\boldsymbol{i}}$ of $\widehat{\boldsymbol{\sigma}}_{\boldsymbol{i}}$ for all $\xi \in \mathbb{R}_+$ and define

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Recall: uniform summability $\implies$ $\Sigma\left[\exp\left(\frac{1}{\hbar}\int_{x_0}^{x} \widehat{s}\,\mathrm{d}x\,/\hbar\right)\right] = \exp\left(\frac{1}{\hbar}\int_{x_0}^{x} \Sigma\big[\widehat{s}\big]\,\mathrm{d}x\right)$

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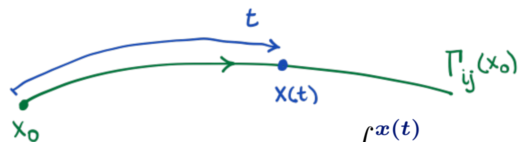

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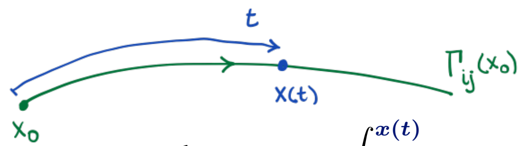

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4. Construct σ_i using the method of successive approximations: define $\{\tau_k(x, \xi)\}$ by

$$\tau_0 := a_0 \,, \qquad \tau_1 := -\int_0^\xi \big(\alpha_0 + a_1 \tau_0\big) \, \mathrm{d}t \,, \qquad \tau_2 := -\int_0^\xi \big(a_1 \tau_1 + \alpha_1 * \tau_0\big) \, \mathrm{d}t \,, \qquad \cdots$$

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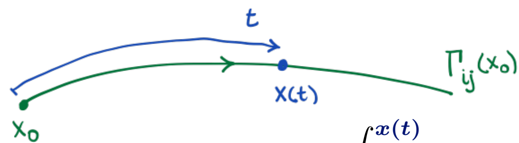


$$\sigma(x, \xi) = a_0 - \int_0^\xi (\text{righthand side}) \Big|_{(\boldsymbol{x(t)}, \xi - t)} \mathrm{d}t \qquad \text{where} \qquad t = \int_{x_0}^{\boldsymbol{x(t)}} \lambda_{ij} \, \mathrm{d}x$$

4. Construct σ_i using the method of successive approximations: define $\{\tau_k(x, \xi)\}$ by

$$\tau_0 := a_0 \,, \qquad \tau_1 := -\int_0^\xi \big(\alpha_0 + a_1 \tau_0 \big) \, \mathrm{d}t \,, \qquad \tau_2 := -\int_0^\xi \big(a_1 \tau_1 + \alpha_1 * \tau_0 \big) \, \mathrm{d}t \,, \qquad \cdots$$

5. **Lemma**: $\boldsymbol{\sigma_i(x, \xi)} := \sum_{k=0}^{\infty} \tau_k(x, \xi)$ is uniformly convergent for all $\xi \in \mathbb{R}_+$, of exponential type, and $\widehat{\sigma}_i$ is its Taylor series at $\xi = 0$ □

§3.2. Proof Outline ($n \geqslant 3$) | skip!

Focus on the equation $(\hbar\partial_x)^{n-1}s + s^n + \ldots = 0 \quad (\blacklozenge)$ and argue as follows.

❶ Rewrite as a nonlinear system: put $y_1 = s, \;\; y_2 = \hbar\partial_x y, \ldots$, and consider

$$\hbar\partial_x y = F(x, \hbar, y)$$

Example (BNR): $\left(\hbar^3\partial_x^3 + 3\hbar\partial_x + 2ix\right)\psi = 0$

$$\rightsquigarrow \quad \hbar^2\partial_x^2 s + 3s\hbar\partial_x s + s^3 + 3s + 2ix = 0$$

$$\rightsquigarrow \quad \hbar\partial_x \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = F(x,y) = -\begin{bmatrix} y_1^2 - y_2 \\ y_1y_2 + 3y_1 + 2ix \end{bmatrix}$$

$\rightsquigarrow$ leading-order solution $y_i^{(0)} = \begin{bmatrix} \lambda_i \\ \lambda_i^2 \end{bmatrix}$

$\rightsquigarrow$ leading-order Jacobian at $y_i^{(0)}$ is $J_i = -\left.\dfrac{\partial F}{\partial y}\right|_{y=y_i^{(0)}} = \begin{bmatrix} 2\lambda_i & -1 \\ \lambda_i^2 + 3 & \lambda_i \end{bmatrix}$

$\rightsquigarrow$ J_i is diagonalisable to $\Lambda_i := \begin{bmatrix} \lambda_i - \lambda_j & \\ & \lambda_i - \lambda_k \end{bmatrix}$

❷ Linearise around the leading-order solution $y_i^{(0)}$ and apply a gauge transformation G to diagonalise the Jacobian J_i:

$$\text{Let } y = y_i^{(0)} + GS \quad \Longrightarrow \quad \hbar\partial_x S + \Lambda_i S = \hbar A_0 + \hbar A_1 S + \underbrace{\cdots}_{\substack{\text{at least quadratic} \\ \text{in } \hbar \text{ or } S}}$$

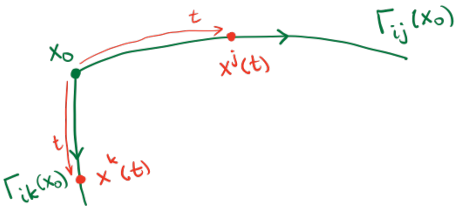
§3.2. Proof Outline ($n \geqslant 3$) | skip!

❸ Apply the Borel transform:

$$\text{Let } \sigma = \mathfrak{B}[S] \quad \Longrightarrow \quad \partial_x \sigma + \Lambda_i \partial_\xi \sigma = \alpha_0 + a_1 \sigma + \alpha_1 * \sigma + \cdots$$

❹ Rewrite as a system of integral equations: $j = 1, \ldots, n-1$

$$\sigma^j(x,\xi) = a_0^j - \int_0^\xi (\text{righthand side})\Big|_{(x^j(t),\,\xi - t)} \mathrm{d}t \qquad \text{where} \qquad t = \int_{x_0}^{x^j(t)} \lambda_{ij}\, \mathrm{d}x$$



❺ Construct σ_i using the method of successive approximations: define $\{\tau_k(x,\xi)\}$ by

$$\tau_0 := a_0\,, \qquad \tau_1 := -\int_0^\xi \big(\alpha_0 + a_1 \tau_0\big)\,\mathrm{d}t\,, \qquad \tau_2 := -\int_0^\xi \big(a_1 \tau_1 + \alpha_1 * \tau_0\big)\,\mathrm{d}t\,, \qquad \cdots$$

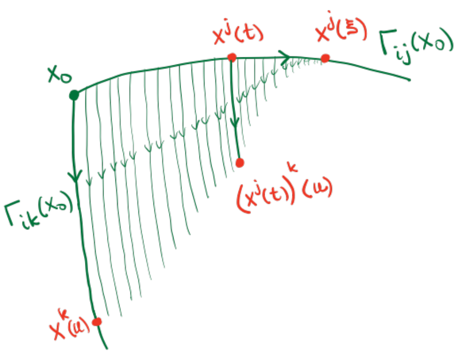
❻ **Lemma 1**: $\sigma_i(x,\xi) := \sum_{k=0}^{\infty} \tau_k(x,\xi)$ is uniformly convergent near $\xi = 0$, and $\widehat{\sigma}_i$ is its Taylor series at $\xi = 0$

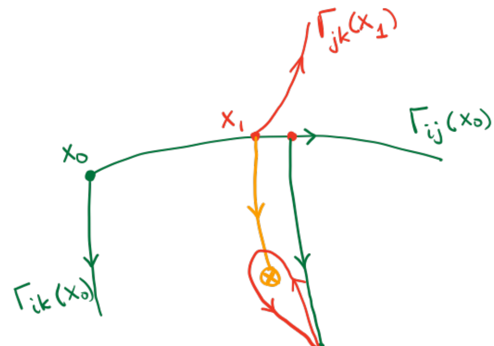
§3.2. Proof Outline ($n \geqslant 3$) | skip!

❻ To analytically continue σ to all $\xi \in \mathbb{R}_+$, carefully examine cross-terms starting in τ_2:

$$\tau_2 := -\int_0^\xi \Big(\underbrace{\boldsymbol{a_1 \tau_1}}_{\begin{bmatrix} \vdots \\ a^j_{11}\tau^1_1 + \ldots + a^j_{1n}\tau^n_1 \\ \vdots \end{bmatrix}} + \alpha_1 * \tau_0 \Big)\,\mathrm{d}t$$

$$\rightsquigarrow \int_0^\xi \int_0^{\xi - t} \tau\Big(\big(x^j(t)\big)^k(u), \xi - t - u \Big)\,\mathrm{d}u\,\mathrm{d}t$$





❼ **Lemma 2**: thanks to the assumption that the (complete) WKB flow is nonsingular, $\sigma(x, \xi)$ admits analytic continuation to $\xi \in \mathbb{R}_+$ of exponential type □

The Geometric WKB Problem

The Geometric WKB Problem

⓿ GIVEN: $(\mathcal{E}, \nabla)$ an oper: $$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

The Geometric WKB Problem

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§4. The WKB Method: Invariant Formulation

The Geometric WKB Problem

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The Geometric WKB Method

❶ Fix a reference pair (W_0, ∇_0) where

- $W_0 : \mathcal{E}'' \to \mathcal{E}$ any **reference splitting**, so $\mathcal{E} \xrightarrow{\sim} \mathcal{E}' \oplus \mathcal{E}''$;
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$$\begin{bmatrix} \mathrm{id} & W \\ 0 & \end{bmatrix} = \begin{bmatrix} \mathrm{id} & S \\ 0 & \mathrm{id} \end{bmatrix} : \quad \begin{array}{ccc} \mathcal{E}' & = & \mathcal{E}' \\ \oplus & \nearrow & \oplus \\ \mathcal{E}'' & = & \mathcal{E}'' \end{array}$$

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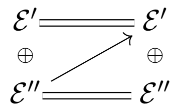

❸ Write $\nabla = \nabla_0 - \phi \quad$ where $\quad \phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$

§4. The WKB Method: Invariant Formulation

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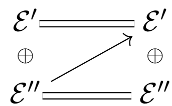

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Its exact solutions yield ***exact WKB flat sections*** for $(\mathcal{E}, \nabla)$

§4. The WKB Method: Invariant Formulation

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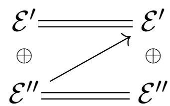

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Remark: $\overset{?}{\Longrightarrow}$ $\quad S \in \mathcal{E}xt^1_X(\mathcal{E}'', \mathcal{E}')$ $\quad \overset{?}{\Longrightarrow}$ $\quad$ cohomological WKB method?

§4. The WKB Method: Invariant Formulation

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⓪ $\hbar^2 \partial_x^2 \psi + q\psi = 0$

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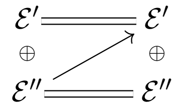

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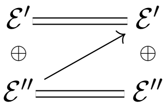

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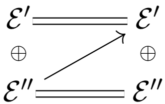

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§4. The WKB Method: Invariant Formulation

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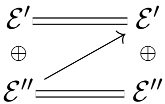

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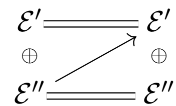

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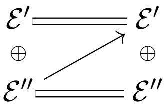

3. Write $\nabla = \nabla_0 - \phi$ where $\phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$.
4. $\mathrm{ad}_{\nabla_0} S - \phi_{11} S + S\phi_{21} S - \phi_{12} + S\phi_{22} = 0$

- Schrödinger equation = 2-nd order $\hbar$-differential operator on $\mathcal{L} := \omega_X^{-1/2}$
- Equivalently, $\hbar$-connection ∇ on the 1-jet bundle $\mathcal{E} := \mathcal{J}^1 \mathcal{L}$
- Oper structure = jet sequence: $0 \longrightarrow \omega_X \otimes \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$
- Reference splitting W_0 is given by choice of coordinate x because

$$\mathcal{E} \xrightarrow[x]{\sim} \left\langle \mathrm{d}x \otimes \mathrm{d}x^{-1/2} \right\rangle \oplus \left\langle \mathrm{d}x^{-1/2} \right\rangle = \mathcal{E}' \oplus \mathcal{E}'' \qquad \text{and} \qquad S = s(x, \hbar)\,\mathrm{d}x$$

- Reference connection $\nabla_0 = \hbar \mathrm{d}$, then $\nabla \underset{\mathrm{loc}}{=} \hbar \mathrm{d} - \begin{bmatrix} 0 & -q \\ 1 & 0 \end{bmatrix} \mathrm{d}x = \nabla_0 - \phi$
- Riccati equation: $\hbar \partial_x s + s^2 + q = 0$

˘◡˘ Thank you for your attention! ˘◡˘