UNIVERSITY OF BIRMINGHAM | SCHOOL OF MATHEMATICS

Invitation to Resurgence

With a View Towards Geometry

Lecture 1

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SCAN FOR LECTURE NOTES

4-7 April 2023

Invitation to Recursion, Resurgence and Combinatorics

Okinawa Institute of Science and Technology (OIST)

Okinawa, Japan

MARIE CURIE ACTIONS

LEVERHULME TRUST

§0. What is this mini-course about?

divergent series and their analytic meaning

How can we promote formal data to analytic data in a natural way?

Brief Plan for the Course:

1. Best example: resurgence of the Euler series
2. Algebras of functions and sectorial neighbourhoods
3. Asymptotic expansions
4. Asymptotic expansions with factorial growth
5. The Borel-Laplace transform
6. Borel resummation
7. The Stokes phenomenon and resurgent series

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alternatively: My Website → Notes

§1. Resurgence of the Euler Series

- **Problem:** find all solutions on the real line of the following ODE

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$$f(x) = Ce^{1/x} + e^{1/x} \int_0^x \frac{e^{-1/u}}{u} \, \mathrm{d}u \qquad \text{for } x > 0,\ C = \text{const} \quad .$$

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- But let's pretend we don't know this, and use the ***power series method***:

power series ansatz $\quad \widehat{f}(x) := \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots$

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substitute into ($\star$) to get a recursion:

$$a_0 = 0,\ a_1 = 1, \quad \text{and} \quad a_{k+1} = -k a_k \quad \text{i.e.} \quad a_{k+1} = (-1)^k k! \qquad \text{for } k \geqslant 1$$

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- Obtain a power series solution called the ***Euler series***:

$$\widehat{\mathrm{Eu}}(x) := \sum_{k=0}^{\infty} (-1)^k k! x^{k+1} = x - x^2 + 2x^3 - 6x^4 + 24x^5 - 120x^6 + \cdots \quad .$$

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* * *

- **Curious historical aside:** why “Euler series”? Clipping from his 1760 paper in *Novi Commentarii academiae scientiarum Petropolitanae*:

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DE
SERIEBVS DIVERGENTIBVS.
Auctore LEON. EVLERO.

§. 13. His praemiſſis neminem fore arbitror, qui me reprehendendum putet, quod in ſummam ſequentis ſeriei diligentius inquiſiuerim:

$$1 - 1 + 2 - 6 + 24 - 120 + 720 - 5040 + 40320 - \text{etc.}$$

quae eſt ſeries a Wallifio hypergeometrica dicta, ſignis alternantibus inſtructa. Haec ſeries autem eo magis

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§. 19. Inveſtigemus nunc etiam analytice huius ſeriei valorem, eam vero in latiori ſenſu accipiamus: ſit igitur

$$s = x - 1x^2 + 2x^3 - 6x^4 + 24x^5 - 120x^6 + \text{etc.}$$

quae differentiata dabit:

$$\frac{ds}{dx} = 1 - 2x + 6xx - 24x^3 + 120x^4 - \text{etc.} = \frac{x-s}{xx}$$

vnde fit $ds + \frac{s\,dx}{xx} = \frac{dx}{x}$, cuius aequationis, ſi e ſuma-

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- This answer is exceptionally simple and beautiful, but comes with two major setbacks:

1. $\widehat{\mathrm{Eu}}(x)$ is divergent and therefore not a true solution!
2. $\widehat{\mathrm{Eu}}(x)$ is at best only a particular solution,
so the power series method has missed most solutions to our ODE!

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Borel resummation legalises this trick!

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The ***Borel sum*** of $\widehat{f}(x)$ is

$$f(x) = \Sigma\left(\widehat{f}(x)\right) := a_0 + \mathfrak{L}\left[\, \varphi(t) \,\right] = a_0 + \mathfrak{L} \circ \mathrm{AnCont} \circ \mathfrak{B}\left[\, \widehat{f}(x) \,\right]$$

Wallis Hypergeometric Series

Question: What is the ‘value’ of

$$\widehat{\mathrm{Eu}}(1) = 1 - 1! + 2! - 3! + 4! - 5! + \cdots = 1 - 1 + 2 - 6 + 24 - 120 + \cdots \quad ?$$

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$$\Sigma\Big(\widehat{\mathrm{Eu}}(1)\Big) = \mathrm{Eu}(1) = \int_0^{\infty} \frac{e^{-t}}{1+t}\, \mathrm{d}t \approx 0.596347362\underline{3}23194\ldots$$

§. 16. Adhibeatur iam haec methodus ad feriem propofitam

A = 1 − 1 + 2 − 6 + 24 − 120 + 720 − 5040 + 40320 − etc.

$$A = \frac{9149852559{,}24}{15343159332{,}90} = 0{,}5963473621237$$

What about $x < 0$?

$$\mathrm{Eu}(x) = \int_0^\infty \frac{e^{-t/x}}{1+t}\,\mathrm{d}t$$

has an obvious problem for $x < 0$: integrand is exponentially growing as $t \to +\infty$

expand our worldview: from now on, x is a complex variable

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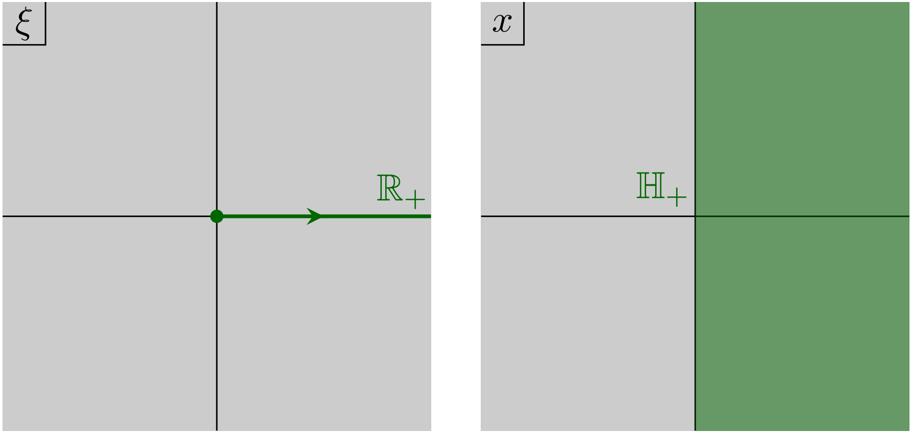

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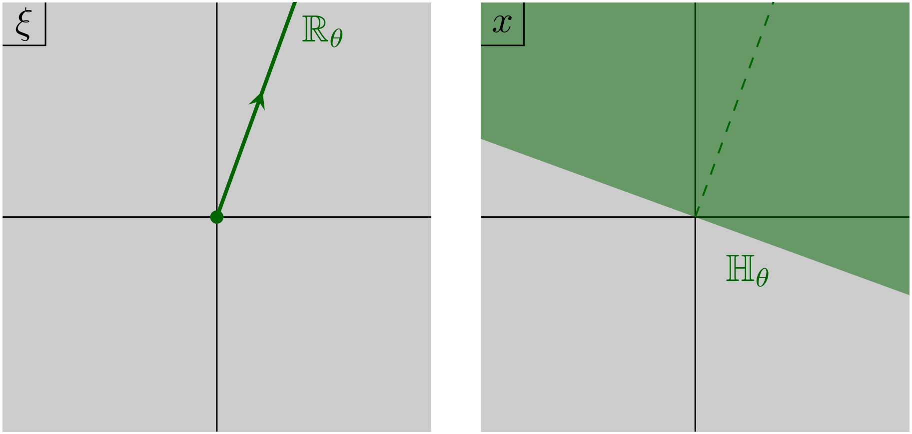
- $\mathfrak{L}_\theta[t](x) := \int_{\mathbb{R}_\theta} te^{-t/x}\,\mathrm{d}t$ is well-defined for all $x \in \mathbb{H}_\theta := \{\mathrm{Re}(e^{-i\theta}x) > 0\}$



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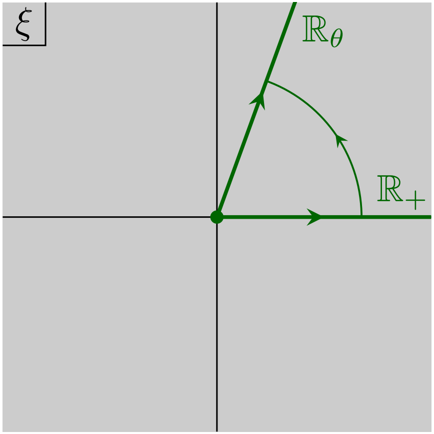
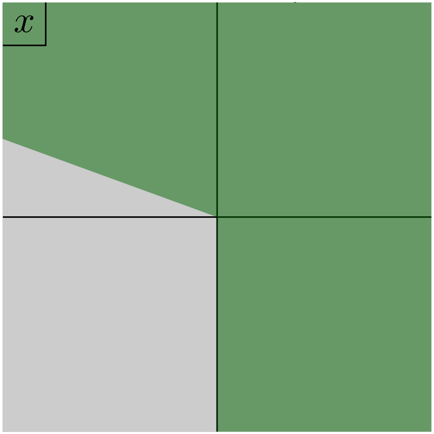
- $\mathfrak{L}_\mathsf{A}[t](x) := \{\mathfrak{L}_\theta[t](x)\}_{\theta\in\mathsf{A}}$ assembles into a holomorphic function on $\bigcup_{\theta\in\mathsf{A}} \mathbb{H}_\theta$

  $\mathsf{A} = (\alpha_-, \alpha_+)$ = arc of directions





Get a particular holomorphic solution for all $x \in \mathbb{C} \setminus \mathbb{R}_-$:

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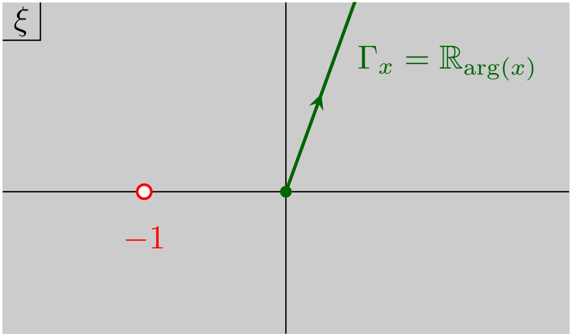

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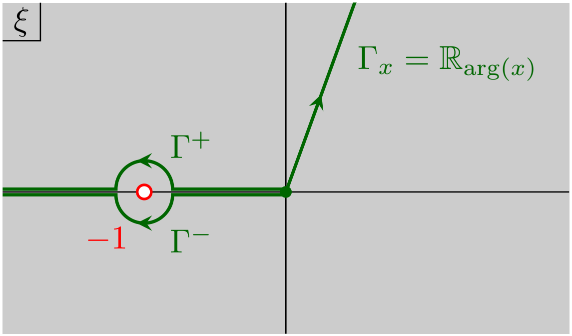
Stokes Phenomenon: they are not the same!



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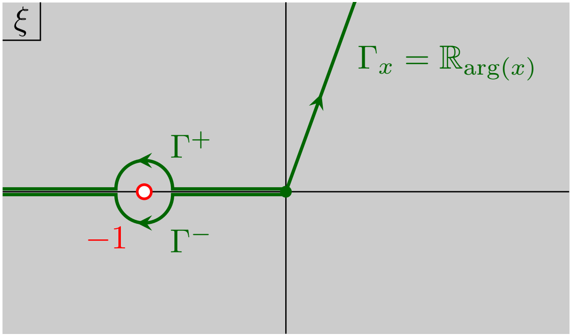
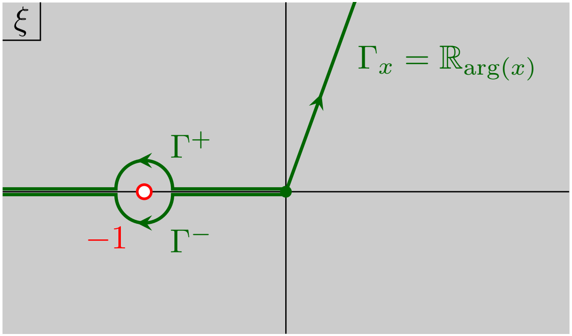


$$\mathrm{Eu}^+(x) - \mathrm{Eu}^-(x) = \oint_{t=-1} \frac{e^{-t/x}}{1+t}\,\mathrm{d}t = 2\pi i \operatorname*{Res}_{t=-1} \left(\frac{e^{-t/x}}{1+t}\,\mathrm{d}t \right) = 2\pi i e^{1/x}$$

§1. Resurgence of the Euler Series

Get a particular holomorphic solution for all $x \in \mathbb{C} \setminus \mathbb{R}_-$:

$$\mathrm{Eu}(x) := \int_{\Gamma_x} \frac{e^{-t/x}}{1+t}\,\mathrm{d}t$$



Consider: $\mathrm{Eu}^{\pm}(x) := \int_{\Gamma^{\pm}} \frac{e^{-t/x}}{1+t}\,\mathrm{d}t$

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So the missing solutions have resurged as residues of the Borel transform!

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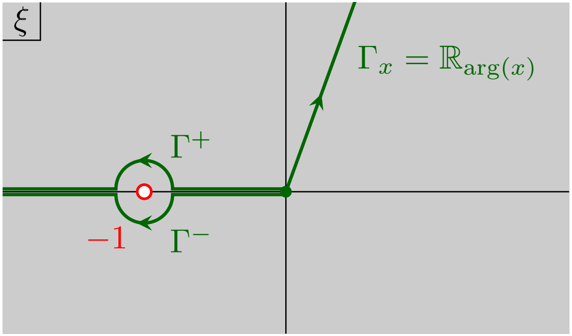

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$$\mathrm{Eu}^{+}(x) - \mathrm{Eu}^{-}(x) = \oint_{t=-1} \frac{e^{-t/x}}{1+t}\,\mathrm{d}t = 2\pi i \operatorname*{Res}_{t=-1} \left(\frac{e^{-t/x}}{1+t}\,\mathrm{d}t \right) = 2\pi i e^{1/x}$$

So the missing solutions have resurged as residues of the Borel transform!

The general solution is the multivalued holomorphic function on $\mathbb{C} \setminus \{0\}$:

$$f(x) = \mathrm{Eu}(x) + Ce^{1/x} = \mathrm{Eu}(x) + C2\pi i \operatorname*{Res}_{t=-1} \left(e^{-t/x}\,\mathrm{eu}(t)\,\mathrm{d}t \right) \qquad C \in \mathbb{C}$$