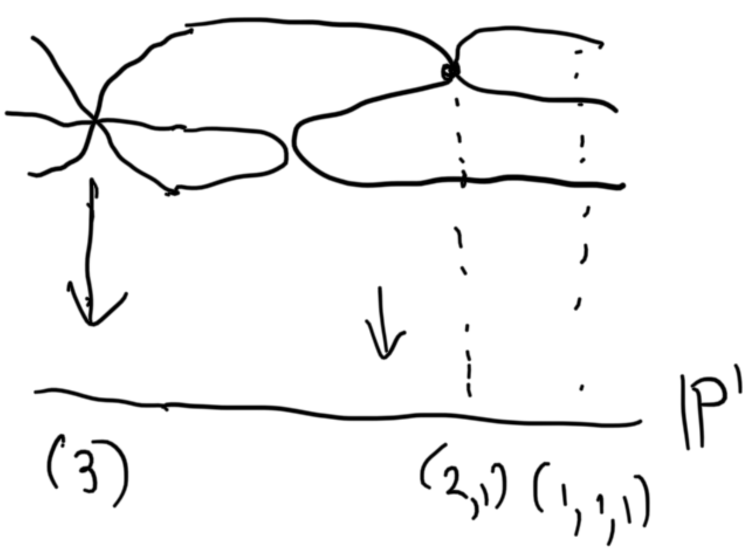
The spin Gromov-Witten/Hurwitz correspondence for $\mathbb{P}^1$

$$\psi(\vartheta) = S(\hbar\partial_\vartheta)\,\vartheta^{r+1}$$

$$\left(\text{if } r = 2s, \quad \psi(\vartheta) = -\psi(-\vartheta) \right)$$

Hurwitz theory: count maps between Riemann surfaces: $f : \Sigma_g \to \mathbb{P}^1$ with given degree d, and given ramification profile over $p_i \in \mathbb{P}^1$

i.e. partitions $\mu_i \vdash d$



Riemann-Hurwitz formula expresses g in terms of d, μ_i

determined by g, and RH

$$h_{g,\mu} = H\left(\mu, (2,1,\dots,1)^{b}\right)$$
$$= \sum_{f \text{ conditions}} \frac{1}{|\mathrm{Aut} f|}$$

$$\omega_{g,n} = d_1 \cdots d_n \sum_{\mu_1,\dots,\mu_n} h_{g,\mu_1,\dots,\mu_n} e^{\sum \mu_i x_i}$$

satisfies TR, as a hypergeometric tau-fn $\psi(\vartheta) = \vartheta$

Gromov-Witten thy constructs a moduli space of maps from Riemann surfaces of genus g to a target X with given homology class

$$\overline{\mathcal{M}}_{g,n}(X, \underset{\substack{\uparrow \\ H_2(X)}}{\beta}) \rightsquigarrow \text{intersection theory}$$

(getting finite numbers)

(gluing finite numbers)

$X = pt$

$$h_{g,\mu} = \boxed{\prod \frac{\mu_i^{\mu_i}}{\mu_i!}} \int_{\overline{\mathcal{M}}_{g,n}(pt)} \frac{\boxed{\Lambda(-1)}}{\prod 1 - \mu_i \psi_i}$$

[ELSV '01], [DGSS '14]

X = curve

More complicated H#

$(2,1,\dots,1) \longrightarrow (r+1, 1, \dots, 1) + \text{l.o.t.}$

completed cycles

[Okounkov, Pandharipande '06]:

H# with completed cycles give

GW of a target curve.

X = surface with $p_g(X) > 0$.

in good cases, lies inside a curve in K_X

We can localise to spin GW in a

spin GW on a curve

A spin structure ϑ on a RS Σ is a square root of K_Σ. There are 2^{2g}.

They have an invariant

$$p(\vartheta) = h^0(\Sigma) \mod 2$$

Spin Hurwitz numbers: $f:(\Sigma,\vartheta)\to(\mathbb{P}^1,\mathcal{O}(-1))$ with specified ramification (which is odd)

$$\sum_{f \text{ cond}} \frac{(-1)^{p(\vartheta)}}{|\mathrm{Aut} f|}$$

No simple ramification!

But there are spin-completed cycles for $r=2s$

$$h^{\vartheta,3}_{g,\mu} = \prod_{i=1}^{n} \frac{\mu_i^{\lfloor \mu_i \rfloor}}{\lfloor \mu_i \rfloor !}\int_{\overline{\mathcal{M}}_{g,n;\frac{1}{2}}} \frac{\Lambda(-1)\Lambda(2)}{\prod_{i=1}^{n} 1-\mu_i\psi_i}$$

[Conj GKL'21, proved by Alexandrov-Shadrin'21]

Using similar techniques as [OP]

Thm [Giacchetto - K. - Lewański - Sauvaget 22]

Spin GW of $\mathbb{P}^1$ is equivalent to spin-completed cycles $\mathcal{M}$#

Thanks!

$$x = z e^{-z^r}$$

$$y = z$$